Universidade de Aveiro Departamento de Matemática
2008

Sérgio da Silva Rodrigues

Métodos da Teoria do Controlo Não Linear em Problemas da Física Matemática

Methods of Nonlinear Control Theory in Problems of Mathematical Physics

Universidade de Aveiro Departamento de Matemática
2008

Sérgio da Silva Rodrigues

Métodos da Teoria do Controlo Não Linear em Problemas da Física Matemática

Methods of Nonlinear Control Theory in Problems of Mathematical Physics

Tese apresentada à Universidade de Aveiro para cumprimento dos requisitos necessários à obtenção do grau de Doutor em Matemática, realizada sob a orientação científica do Doutor Andrey V. Sarychev, Professor Catedrático do DiMaD da Universidade de Florença, Itália e do Doutor Andrey A. Agrachev, Professor Catedrático da SISSA, Trieste, Itália.

Thesis presented to the University of Aveiro in partial fulfilment of the requirements for the PhD degree in Mathematics, elaborated under the supervision of Professor Andrey V. Sarychev from the DiMaD of the University of Florence, Italy and of Professor Andrey A. Agrachev from SISSA-ISAS, Trieste, Italy.

Apoio financeiro do CNRS – Centre National de la Recherche Scientifique (França) - no âmbito do Quadro Comunitário «Improving the Human Research Potential and the Socio-Economic Knowledge Base». Marie Curie CTS (Maio 2002 – Abril 2003).

Apoio financeiro da FCT no âmbito do III Quadro Comunitário de Apoio, comparticipado pelo Fundo Social Europeu e por fundos nacionais do MCTES. Bolsa no âmbito do POS_C – Desenvolver Competências – Medida 1.2 (Novembro 2003 – Outubro 2007)

Dedico este trabalho os meus pais:
To my parents:

João Marques Rodrigues e Maria Virgínia Luz Silva Rodrigues

o júri

presidente

Doutor Casimiro Adrião Pio
Professor Catedrático da Universidade de Aveiro

Doutor Andrey Sarychev
Professor Catedrático da Universidade de Florença – Itália (Orientador)

Doutor Andrey Agrachev
Professor Catedrático da International School for Advanced Studies – Trieste, Itália (Co - Orientador)

Doutora Maria de Fátima da Silva Leite
Professora Catedrática da Faculdade de Ciências e Tecnologia da Universidade de Coimbra

Doutor António Manuel Rosa Pereira Caetano
Professor Associado com Agregação da Universidade de Aveiro

Doutora Adélia da Costa Sequeira dos Ramos Silva
Professora Associada com Agregação do Instituto Superior Técnico da Universidade Técnica de Lisboa

Doutor Vítor Manuel Carvalho das Neves
Professor Associado da Universidade de Aveiro

agradecimentos

Aos meus orientadores, Prof. Andrey Sarychev e Prof. Andrey Agrachev: a orientação científica que me proporcionaram, marcada pela disponibilidade, paciência e apoio. Quando eu encontrava alguns obstáculos era através de discussões inspiradoras com eles que aprendia como ultrapassar os mesmos obstáculos. Sem a ajuda deles este trabalho não seria possível. Foi devido à vasta cultura matemática deles que melhorei o meu modo de fazer investigação em Matemática e de conhecer mais relações entre as várias áreas da Matemática. Do ponto de vista humano, senti que foram meus amigos.

À Prof. Ana Breda, ao Prof. António Caetano e ao Prof. Vítor Neves, do Departamento de Matemática da Universidade de Aveiro: o terem acreditado, e referenciado, nos primeiros passos deste trabalho, que eu poderia fazer investigação em Matemática.

À Universidade de Aveiro (em particular ao Departamento de Matemática) e à SISSA (em particular ao *Settore di Analisi Funzionale*): o terem aceitado este projecto. Ainda à SISSA, a oportunidade que tive de seguir alguns cursos, leccionados por Professores de renome internacional, e o uso das instalações e serviços os quais me proporcionaram todo o apoio para a minha pesquisa.

Ao *Marie Curie Control Training Site*: a oportunidade de começar a trabalhar em Teoria do Controlo - bolsa Ref. HPMT-GH-01-00278-12 (Maio 2002 - Abril 2003) - e de participar em algumas conferências nacionais e internacionais onde tive a oportunidade de conhecer outros matemáticos e de com esses trocar ideias.

À Fundação para a Ciência e a Tecnologia: a bolsa no âmbito do POS_C - Ref. SFRH/BD/13554/2003 (Novembro 2003 - Outubro 2007).

Aos meus pais João M. Rodrigues e Maria Virgínia L. S. Rodrigues; irmão Helder S. Rodrigues e; irmã Arlete S. Rodrigues: o amor e apoio familiar. Gostaria também, neste momento, de lembrar a minha irmã Maria de Fátima que não cheguei a conhecer mas, foi-me ensinado, \'e um Anjo que olha por mim do Céu.

acknowledgements

To my supervisors, Prof. Andrey Sarychev and Prof. Andrey Agrachev: the provided scientific supervision, marked by the availability, the patience and the support. When I have found some obstacles it was trough inspiring discussions with them that I learned how to pass over the same obstacles. Without their help this work would not be possible. It has been also thanks to their mathematical culture that I could improve my way of doing research in mathematics and to see more connections between several branches of Mathematics. From the human point of view, I felt they were my friends.

To Prof. Ana Breda, to Prof. António Caetano and to Prof. Vítor Neves, of the Department of Mathematics of the University of Aveiro: the believing, and referring, in the first steps of this work, that I could do research in Mathematics.

To the University of Aveiro (in particular the Department of Mathematics) and to SISSA (in particular the *Settore di Analisi Funzionale*): for accepting this project. Yet, to SISSA the opportunity given me to follow some courses, lectured by Professors of international reputation, and the use of the installations and services whose provided me all sustain to my research.

To the Marie Curie Control Training Site: the opportunity to start working in Control Theory - fellowship Ref. HPMT-GH-01-00278-12 (May 2002 - April 2003) - and to participate in some national and international conferences.

To the *Fundação para a Ciência e a Tecnologia*: the scholarship in the scope of the POS_C project - Ref. SFRH/BD/13554/2003 (November 2003 - October 2007).

To my parents João M. Rodrigues and Maria Virgínia L. S. Rodrigues; brother Helder S. Rodrigues and; sister Arlete S. Rodrigues: the familiar love and support. At this moment I would like also to remember my sister Maria de Fátima I did not know but, I was taught, is an Angel looking after me from Heaven.

palavras-chave Fluido incompressível, sistema de Navier-Stokes 2D, controlabilidade.

resumo

Consideramos a equação de Navier-Stokes num domínio bidimensional e estudamos a sua controlabilidade aproximada e a sua controlabilidade nas projecções em subespaços de campos vectoriais de dimensão finita. Consideramos controlos internos que tomam valores num espaço de dimensão finita. Mais concretamente, procuramos um subespaço de campos vectoriais de divergência nula de dimensão finita de tal modo que seja possível controlar aproximadamente a equação, através de controlos que tomam valores no mesmo subespaço. Usando algumas propriedades de continuidade da equação nos dados iniciais, nomeadamente a continuidade da solução quando o controlo varia na chamada métrica relaxada, reduzimos os resultados em controlabilidade à existência de um chamado conjunto saturante. Consideramos ambas as condições de fronteira do tipo Navier e Dirichlet homogéneas. Damos alguns exemplos de domínios e respectivos conjuntos saturantes. No caso especial das condições de fronteira do tipo Lions - um caso particular das condições do tipo Navier - através de uma técnica envolvendo perturbação analítica de métricas, transferimos a chamada controlabilidade nas projecções em espaços coordenados de dimensão finita de uma métrica para (muitas) outras.

keywords Incompressible fluid, 2D Navier-Stokes system, controllability.

abstract We consider the Navier-Stokes equation on a two-dimensional domain and study its approximate controllability and its controllability on projections onto finite-dimensional subspaces of vector fields. We consider body controls taking values in a finite-dimensional space. More precisely we look for a finite-dimensional subspace of divergence free vector fields that allow us to control approximately the equation using controls taking values in that subspace. Using some continuity properties of the equation on the initial data, namely the continuity of the solution when the control varies in so-called relaxation metric, we reduce the controllability issues to the existence of a so-called saturating set. Both Navier and no-slip boundary conditions are considered. We present some examples of domains and respective saturating sets. For the special case of Lions boundary conditions - a particular case of Navier boundary conditions - trough a technique involving analytic perturbation of metrics, we transfer so-called controllability on observed coordinate space from one metric to (many) other.

Methods of Nonlinear Control Theory in Problems of Mathematical Physics

Sérgio S. Rodrigues

"O frati", dissi, "che per cento milia
perigli siete giunti a l'occidente,
a questa tanto picciola vigilia
d'i nostri sensi ch'è del rimanente
non vogliate negar l'esperïenza,
di retro al sol, del mondo sanza gente.
Considerate la vostra semenza:
fatti non foste a viver come bruti,
ma per seguir virtute e canoscenza".

In "Divina Commedia" by Dante Alighieri. (Inferno, Canto XXVI, vv. 112-120).
Edition: Newton Compton editori s.r.l., Roma, 1993.

Contents

Introduction

Navier-Stokes equations are equations governing the motion of a fluid like air, water or oil. They appear, sometimes coupled with other equations, in the study of several phenomena like hydraulics, meteorology or plasma physics and can be derived from the principles of conservation of mass and momentum in mechanics (see [24]). The denomination Navier-Stokes come from the mathematicians Claude-Louis Navier and George Gabriel Stokes.

There are two classical representations of the Navier-Stokes equations: the Lagrangean representation and the Eulerian representation. The former describes the state of a fluid "particle" at a given time from its initial state position while, the latter describes the velocity of the fluid particle at a give state position particle and given time from the initial velocities at each particle. Here we consider the Eulerian representation.

The fluids considered in this work are viscous, incompressible and homogeneous, i.e., our fluid presents a kinematic viscosity, any set of particles fills a set of the same volume at every instant of time and the density of the fluid is homogeneous.

The Navier-Stokes equations, governing a fluid in a domain (a container) Ω, are

$$u_t + (u \cdot \nabla)u + \nabla p = -\nu \Delta u + F(x);\,^{1}$$
$$\nabla \cdot u = 0;$$

where $u = u(x, t)$ is the *velocity of the fluid "particle"* $x \in \Omega$ at time t; $\nu > 0$ is the *kinematic viscosity*; $p = p(x, t)$ is the *pressure* and F is an *external force* to the system.

$u_t := \frac{\partial u}{\partial t}$; $-\nu\Delta u$ is called the *viscosity term* and the only nonlinear term of the equation — $(u \cdot \nabla)u$ — is called the *inertial term*. Our vector fields are divergence free — $\nabla \cdot u = 0$ — due to incompressibility.

Given an initial condition $u(x, 0) = u_0(x)$ for any state particle x, we may ask for the existence and uniqueness of a solution of the Navier-Stokes system satisfying the given initial condition.

A natural way, we adopt here, to study the Navier-Stokes equations is to study its evolution on subspaces of Sobolev spaces. Many studies have been done in this setting about existence, uniqueness, regularity and continuity on initial data — see for example the works by R. Temam in [56]; by P. Constantin and C. Foias in [19] and by A.V. Babin and M.I. Vishik in [13]. For works in the setting of Riemannian manifolds we refer to the works by A. A. Il'in in [30] and by V. Priebe in [42].

Results on continuity of the solution on initial data are important in the study of controllability of the Navier-Stokes system that is the subject we are mainly interested in.

We consider one more external force $v = v(x, t)$ we are able to change, in other words, v will be our (body) control. We are interested in the case this control is degenerated.

[1] We use the notation $\Delta = -\left(\frac{\partial^2}{x_1^2} + \frac{\partial^2}{x_2^2}\right)$.

The task is to drive the system from the initial position u_0 to another given position u_1 in some given time $T > 0$. Is it possible to do for any triple (u_0, u_1, T)? Since the Navier-Stokes equations are related to several phenomena in Nature, controllability studies are an important task so, it it not surprising that many studies on the subject have been done before — see for example the works by A. V. Fursikov and O. Yu. Imanuvilov in [26], by J.-M. Coron and A. V. Fursikov in [20]; by A. Shyrikian in [51, 52]; by A. A. Agrachev and A. V. Sarychev in [4, 5] and by J.-L. Lions and E. Zuazua in [36, 37].

This work follows the study done by A. Agrachev and A. Sarychev in [4] where a new method, based on the bilinear term of the equation, has been invented and has led to new controllability results concerning the two-dimensional Navier-Stokes system governing the motion of a fluid in the torus $\mathbb{T}^2$. This method have been used to study controllability in three-dimensional case by A. Shyrikian in [51, 52]. We use it in the case the fluid fills a two-dimensional domain. Boundary conditions as no-slip and Navier are considered.

The method points to controllability by means of degenerate forcing, i.e., we look for a finite-dimensional subspace of vector fields such that the equation is (approximately) controllable by means of controls taking values in that subspace. Tools from Geometric Control Theory are involved; for more details on these tools we refer to the books by A. Agrachev and Yu. Sachkov [3] and, by V. Jurdjević [32].

Two-dimensional Navier-Stokes equation may be seen as a good approximation for three-dimensional one on thin three-dimensional containers like thin films. Some questions concerning existence, uniqueness and regularity of the solutions are fixed for two-dimensional case and still open for three-dimensional case. On the other side the two-dimensional equation can be reduced to a scalar equation for the so-called vorticity so, the study of two-dimensional case presents more flexibility.

The Navier-Stokes equation may be seen as an evolutionary equation on the subspace of divergence free vector fields: projecting each term of the equation onto the subspace of divergence free vector fields we obtain the equation

$$u_t + Bu = -\nu(A - C)u + F + v;$$

where Bu is the projection of the bilinear term and $-\nu(A-C)u$ is the projection of the viscosity term (we write $A-C$ for the "projection" of Δ because we want some specific properties for A and sometimes the exact projection $A-C$ do not have them). We suppose our external forces are divergence free, otherwise we should take their projections. As we see the gradient of the pressure, that is orthogonal to the space of divergence free vector fields, is a kind of correction of the equation: not all the terms of the equation are divergence free so the gradient of the pressure corrects that fact.

This work is organized as follows:

In chapter 1 we consider Navier-Stokes evolutionary equations and derive some results on continuity of the equation on initial data, namely the continuous dependence of the solution when the control varies in so-called relaxation metric.

In chapter 2 we introduce the notion of saturating set of vector fields; the existence of such a set will be the sufficient condition for the controllability results.

Controllability is treated in chapter 3. Under the hypothesis of existence of a saturating set we conclude both approximate controllability in $L^2(T\Omega)$-norm and controllability on observed finite-dimensional component, i.e., we can observe (exact) controllability if we look at the projections of the solutions onto a given finite-dimensional space of vector fields.

In chapters 4 and 5 we consider the fluid fills, respectively, Euclidean and Riemannian domains. For no-slip and Navier boundary conditions the respective evolutionary equations suit the properties we need to derive the results on controllability.

In chapter 6 we present the the cases the domain is the Torus, the Sphere, the Rectangle and the Hemisphere; and respective examples of saturating sets. In the last two we consider Navier boundary conditions.

Controllability of finite-dimensional Galerkin approximations of the equation, are studied in chapter 7 under the existence of a "special" saturating set.

Under Lions boundary conditions, in chapter 8, we derive some partial results by a technique involving perturbation of metrics in a given compact Riemannian manifold; it turns out that controllability on observed coordinate space may be transferred from one metric to "many other" metrics. For that we ask the boundaries of the domain to be analytic. As a corollary if we have controllability on observed coordinate space for a simply connected bounded plane domain then we have it for many other analytic plane domains.

Some of the results obtained, in collaboration with A. Agrachev and A. Sarychev, during this work time have been published either in Journal or proceedings: [44, 45, 46, 47, 48, 49]; closely related are the works [5, 6] by A. Agrachev and A. Sarychev and; in the 3D case, the works [51, 52] by A. Shirikyan.

Chapter 1

Navier-Stokes evolutionary equations

In this chapter we study some continuity properties of evolutionary equations of Navier-Stokes type. It this way we may treat simultaneously several boundary conditions, just asking some common properties for the operators appearing in the equation. So we do not speak about specific boundary conditions, for the moment.

1.1 The operators and spaces

Let us fix two Hilbert spaces V and H with

$$V \subset H \quad \text{densely, continuously and compactly.} \tag{1.1}$$

Let us denote the scalar products and norms of these spaces by

$$((\cdot,\, \cdot)),\ \|\cdot\| \text{ for } V; \quad (\cdot,\, \cdot),\ |\cdot| \text{ for } H.$$

1.1.1 The linear operator A

We denote the canonical isomorphism between V and its dual V', associated to $((\cdot,\, \cdot))$ by A, i.e., $A : V \to V'$

$$((u,\, v)) =: \langle Au,\, v\rangle_{V',V}.$$

It turns out that the inclusions

$$V \subset H \subset V'$$

(identifying H with its dual) are both continuous, dense and compact and, for $v \in V$ and $u \in H$, we have $\langle u,\, v\rangle_{V',V} = (u, v)$.

Moreover we may define the domain $D(A)$ of the operator A in H as

$$D(A) := \{u \in V \mid Au \in H\}$$

and consider A as an unbounded linear operator in H. The operator A is strictly positive, i.e., $(Au,\, u) = \|u\|^2 > 0$, for all $u \in D(A) \setminus \{0\}$.

We endow $D(A)$ with the scalar product $(u,\, v)_{[2]} := (Au,\, Av)$ and respective norm $|u|_{[2]} = |Au|$. A turns out to be an isomorphism from $D(A)$ onto H.

Since the injection $V \to H$ is compact the operator A^{-1} may be considered as a compact operator in H. We infer (see for example [11, ch. 11, th. 2]) that there exists a complete orthonormal basis

$$\mathcal{W} := \{W_j \mid j \in \mathbb{N}_0\}$$

where $\mathbb{N}_0 := \mathbb{N} \setminus \{0\}$.

$$A^{-1}W_j = \mu_j W_j, \quad j \in \mathbb{N}_0;$$
$$\mu_j \text{ a decreasing sequence;} \quad \mu_j \to 0 \quad \text{as} \quad j \to \infty$$

It is clear that for each $j \in \mathbb{N}_0$ we have $W_j \in D(A)$ and, setting $k_j = \mu_j^{-1}$ we obtain

$$AW_j = k_j W_j, \quad j \in \mathbb{N}_0;$$
$$0 < k_1 \leq k_2 \leq \cdots \leq k_j \leq \cdots; \quad k_j \to \infty \quad \text{as} \quad j \to \infty.$$

The family $\mathcal{W}$ is orthogonal in H, V and $D(A)$:

$$\begin{aligned}
&(W_j, W_i) = \delta_{ji};\\
&((W_j, W_i)) = \langle AW_j, W_i\rangle_{V',V} = (AW_j, W_i) = k_j\delta_{ji};\\
&(W_j, W_i)_{[2]} = (AW_j, AW_i) = k_j k_i (W_j, W_i) = k_j^2\delta_{ji}
\end{aligned}$$

We also have

$$(W_j, W_i)_{V'} = (W_j, A^{-1}W_i)_{V',V} = k_i^{-1}\delta_{ji}.$$

We may also define the powers $A^s : D(A^s) \to H$ of $A : D(A) \to H$, for $s \in \mathbb{R}$ (see [11, ch. 11]). For $s > 0$, A^s is an unbounded self adjoint operator in H with dense domain $D(A^s) \subseteq H$; A^s is strictly positive and $D(A^s)$ is endowed with the scalar product $(u, v)_{D(A^s)} := (A^s u, A^s v)$ and norm $|u|_{D(A^s)} := |A^s u|$. Moreover A^s is an isomorphism from $D(A^s)$ onto H.

For $s = 1$ we recover $D(A)$ and for $s = \frac{1}{2}$ we have $D(A^{1/2}) = V$.

For $s = 0$ we put $A^0 = I$, $D(A^0) = H = H'$. For $s > 0$ we put

$$D(A^{-s}) := \text{ dual of } D(A^s),$$

so A^s can be extended to an isomorphism from H onto $D(A^{-s})$. We endow $D(A^{-s})$ with the scalar product $(u, v)_{D(A^{-s})} := (A^{-s}u, A^{-s}v)$ and norm $|u|_{D(A^{-s})} := |A^{-s}u|$.

For $s = -\frac{1}{2}$ we recover V'.

For every $s_1 > s_0$ the inclusion $D(A^{s_1}) \subset D(A^{s_0})$ is dense, continuous and compact. The map $A^{s_1 - s_0}$ is an isomorphism of $D(A^{s_1})$ onto $D(A^{s_0})$.

For $s \geq 0$ we have the characterizations:

$$(u, v)_{D(A^s)} = \sum_{j=1}^{+\infty} k_j^{2s}(u, W_j)(v, W_j)$$
$$|u|^2_{D(A^s)} = \sum_{j=1}^{+\infty} k_j^{2s}(u, W_j)^2$$

and then

$$D(A^s) = \left\{ u \in H \mid \sum_{j=1}^{+\infty} k_j^{2s}(u, W_j)^2 < +\infty \right\}.$$

For $s < 0$, $D(A^s)$ is the completion of H for the norm

$$\left(\sum_{j=1}^{+\infty} k_j^{2s}(u,\,W_j)^2\right)^{\frac{1}{2}}.$$

For $s \in \mathbb{R}$ if $u = \sum_{j=1}^{+\infty}(u,\,W_j)W_j \in D(A^s)$, then $A^s u \in H$ is given by

$$A^s u = \sum_{j=1}^{+\infty} k_j^s(u,\,W_j)W_j.$$

1.1.2 The bilinear operator B

We fix a trilinear form b

$$\begin{aligned} H^3 &\to \mathbb{R}\cup\{\infty\} \\ (u,\,v,\,w) &\mapsto b(u,\,v,\,w) \end{aligned}$$

where $\mathbb{R}\cup\{\infty\}$ means that it may exist some triples where the form is not defined (its value is not real); for b we suppose to have the estimates

$$|b(u,\,v,\,w)| \le C_1 K$$

where C_1 is a constant and K is one of the following products

$$\|u\|\|v\|\|w\| \qquad \text{for } u,\,v,\,w \in V; \tag{1.2}$$

$$|u|^{\frac{1}{2}}\|u\|^{\frac{1}{2}}\|v\|^{\frac{1}{2}}\|v\|_{[2]}^{\frac{1}{2}}|w| \qquad \text{for } u \in V,\, v \in D(A),\, w \in H, \tag{1.3}$$

$$|u|^{\frac{1}{2}}\|u\|_{[2]}^{\frac{1}{2}}\|v\||w| \qquad \text{for } u \in D(A),\, v \in V,\, w \in H, \tag{1.4}$$

$$|u|\|v\||w|^{\frac{1}{2}}\|w\|_{[2]}^{\frac{1}{2}} \qquad \text{for } u \in H,\, v \in V,\, w \in D(A), \tag{1.5}$$

$$|u|^{\frac{1}{2}}\|u\|^{\frac{1}{2}}\|v\||w|^{\frac{1}{2}}\|w\|^{\frac{1}{2}} \qquad \text{for } u,\,v,\,w \in V. \tag{1.6}$$

In particular, by (1.2) we have that the form b is continuous in V^3 and, for each pair $(u,\,v) \in V^2$ we may define the operator $B(u,\,v) \in V'$ by

$$\begin{aligned} B(u,\,v) : V &\to \mathbb{R} \\ w &\mapsto \langle B(u,\,v),\,w\rangle_{V',V} = b(u,\,v,\,w) \end{aligned}$$

and we set $Bu = B(u) := B(u,\,u) \in V', \quad \forall u \in V$.

Another property we ask for b is that fixed the first variable in V, the form b results skew-symmetric in the last two variables, i.e.,

$$\forall u \in V\,[\,b(u,\,v,\,w) = -b(u,\,w,\,v)\,], \quad b \text{ defined in both } (u,\,v,\,w) \text{ and } (u,\,w,\,v).$$

In particular for $u \in V$, we have

$$b(u,\,v,\,v) = 0, \quad b \text{ defined in } (u,\,v,\,v).$$

Remark 1.1.1. *A form b being skew-symmetric in the last two variables in the product $X \times Y \times Y$ and continuous in $X \times Y \times Z$ with Y dense in Z, may be extended continuously to $X \times Z \times Y$: set a sequence $(w^n)_{n\in\mathbb{N}} \in Y$ converging to w in Z-norm; the sequence $b(u, w^n, v) = -b(u, v, w^n)$ converges to $-b(u, v, w)$. Hence we may define $b(u, w, v) := -b(u, v, w)$ and we obtain a continuous extension of b to $X \times Z \times Y$; the extension remains skew-symmetric in the last two variables.*

Analogously, if b is skew-symmetric in the last two variables in the product $X \times Y \times Y$ and continuous in $X \times Z \times Y$, with Y dense in Z, it may be extended in the same way to $X \times Y \times Z$.

Therefore we also have all the estimates obtained from the above, by "symmetry" in the last two variables.

1.1.3 The linear operator C

Treating different kinds of domains and of boundary conditions we may need to write the projection of the viscosity term as $-\nu(A - C)u$ to have the desired properties for the linear operator A. We consider the case C is a linear operator

$$C : V \to H$$

with the properties

$$\begin{aligned}(Cu, v) &= (u, Cv), \quad u, v \in V;\\ (Cu, v) &\le K\|u\||v|, \quad u \in V, v \in H.\end{aligned}$$

1.2 Useful tools

In the study of existence, uniqueness and regularity of the solutions of the Navier-Stokes equation, we need some classical results. Now we present some of them and indicate where the proof may be found.

Lemma 1.2.1 (Gronwall inequality. [55] ch. III subsection 1.1.3)**.** *Let g, h, y, $\frac{dy}{dt}$ be locally integrable functions satisfying*

$$\frac{dy}{dt} \le gy + h \quad \text{for } t \ge t_0. \tag{1.7}$$

Then

$$y(t) \le y(t_0) \exp\left(\int_{t_0}^t g(\tau)\, d\tau\right) + \int_{t_0}^t h(s) \exp\left(-\int_t^s g(\tau)\, d\tau\right) ds, \quad t \ge t_0.$$

Lemma 1.2.2 (Young inequality. [54] ch. 3)**.** *Given a, b, $\varepsilon > 0$, $1 < p < +\infty$, we have*

$$ab \le \varepsilon a^p + C^Y_{\varepsilon,p} b^{p'}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$ and $C^Y_{\varepsilon,p} = \dfrac{p-1}{\left(p^{p'}\right)\left(\varepsilon^{\frac{1}{p-1}}\right)} < \varepsilon^{-\frac{1}{p-1}}$.

Lemma 1.2.3 ([56] section 3.1). *Let X be a Banach space with dual X' and let u and g be two functions belonging to $L^1(a,\,b,\,X)$. Then the following three conditions are equivalent*

1. *u is a.e. equal to a primitive function of g,*
$$u(t) = \xi + \int_0^t g(s)\,ds, \quad \xi \in X,\ a.e.\, t \in [a,\,b];$$

2. *For each test function $\phi \in \mathcal{D}(]a,\,b[)$,*
$$\int_a^b u(t)\phi'(t)\,dt = -\int_a^b g(t)\phi(t)\,dt \qquad \left(\phi' = \frac{d\phi}{dt}\right);$$

3. *For each $\eta \in X'$,*
$$\frac{d}{dt} < u,\,\eta > = < g,\,\eta >$$
in the scalar distribution sense, on $]a,\,b[$

In particular if 1–3 is satisfied, u is a.e. equal to a continuous function $[a,\,b] \to X$.

Theorem 1.2.4 ([56] section 3.2). *Let $X_0 \subset X \subset X_1$ be Hilbert spaces with both inclusions being continuous and the first one being, in addiction, compact. Then for any bounded set K and any $\gamma > 0$ the injection of $\mathcal{H}^\gamma_K(\mathbb{R},\,X_0,\,X_1)$ into $L^2(\mathbb{R},\,X)$ is compact. Here*
$$\mathcal{H}^\gamma_K(\mathbb{R},\,X_0,\,X_1) = \{u \in \mathcal{H}^\gamma(\mathbb{R},\,X_0,\,X_1) \mid \operatorname{supp} u \subset K\}$$
where
$$\mathcal{H}^\gamma(\mathbb{R},\,X_0,\,X_1) = \{v \in L^2(\mathbb{R},\,X_0) \mid \ D^\gamma_t v \in L^2(\mathbb{R},\,X_1)\}.$$
is the Hilbert space which norm $|\cdot|_{\mathcal{H}^\gamma(\mathbb{R},\,X_0,\,X_1)}$ defined by
$$|v|^2_{\mathcal{H}^\gamma(\mathbb{R},\,X_0,\,X_1)} = |v|^2_{L^2(\mathbb{R},\,X_0)} + ||\tau|^\gamma \hat{v}|_{L^2(\mathbb{R},\,X_1)}.$$
By $D^\gamma_t v$ we mean the derivative (in t) of order γ of v. Its Fourier Transform is given by $(i\tau)^\gamma \hat{v}$, i.e.,
$$\widehat{D^\gamma_t v(\tau)} = (i\tau)^\gamma \hat{v}(\tau).$$

We recall that the Fourier Transform $\mathcal{F}[f] = \hat{f}$ of a continuous, absolutely integrable function f in $\mathbb{R}^m$ is (or may be) defined as $\hat{f}(\xi) := (2\pi)^{-m/2}\int_{\mathbb{R}^m} e^{-i\xi\cdot x} f(x)\,dx$. For tempered distributions $f \in \mathcal{S}'(\mathbb{R}^m)$ we define $\mathcal{F}[f] = \hat{f}$ by the relation $\langle \mathcal{F}[f],\,\phi\rangle = \langle f,\,\mathcal{F}^{-1}[\phi]\rangle$, for all $\phi \in \mathcal{S}(\mathbb{R}^m)$, where $\mathcal{F}^{-1}[\phi](x) := (2\pi)^{-m/2}\int_{\mathbb{R}^m} e^{i\xi\cdot x}\phi(\xi)\,d\xi$. Properties of the Fourier Transform may be found, for example, in [43].

Below we shall use the symbols: $\rightharpoonup$ for weak convergence; $\rightharpoonup_*$ for weak-star convergence and, $\to$ for strong convergence.

Lemma 1.2.5 ([14] section III.6). *Let E be a reflexive Banach space and let (x_n) a bounded sequence in E. Then there is a subsequence $(x_{\sigma(n)})$ of (x_n) such that $x_{\sigma(n)} \rightharpoonup x$, for some $x \in E$.*

Lemma 1.2.6 ([14] section III.6). *Let E be a separable Banach space and let (f_n) a bounded sequence in E'. Then there is a subsequence $(f_{\sigma(n)})$ of (f_n) such that $f_{\sigma(n)} \rightharpoonup_* f$, for some $f \in E'$.*

Lemma 1.2.7 ([35] ch. 3 theorem 3.1). *Let*

$$u \in L^2(0,\, T,\, D(A^{s_1})) \;\;\&\;\; u' \in L^2(0,\, T,\, D(A^{s_0}))$$

with $s_1 > s_0$; then u is a.e. equal to a continuous function from $[0,\, T]$ into $D\Big(A^{\frac{s_1+s_0}{2}}\Big)$.

Moreover (see [56, section 3.1]) in the case $-s_0 = s_1 > 0$ the equality

$$\frac{d}{dt}|u|^2 = 2 < u',\, u > \tag{1.8}$$

holds in the distribution sense on $(0,\, T)$. [1]

Remark 1.2.1. *The space $D\Big(A^{\frac{s_1+s_0}{2}}\Big)$ may be seen as an interpolation space between $D(A^{s_1})$ and $D(A^{s_0})$. With the notations in [35] for $\theta \in [0,\, 1]$:*

$$[D(A^{s_1}),\, D(A^{s_0})]_\theta = D\Big((A^{s_1-s_0})^{1-\theta}\Big);$$

where $\tilde{A} := A^{s_1-s_0}$ is to be seen as an operator from $D(A^{s_1})$ to $D(A^{s_0})$ with $D(\tilde{A}) = D(A^{s_1})$ and $D(\tilde{A}^0) = D(A^{s_0})$. Then by $\tilde{A}^{1-\theta}D(\tilde{A}^{1-\theta}) = D(A^{s_0})$, we have $D(\tilde{A}^{1-\theta}) = \tilde{A}^{\theta-1}D(A^{s_0}) = \tilde{A}^{\theta}D(A^{s_1})$, i.e.,

$$D((A^{s_1-s_0})^{1-\theta}) = (A^{s_1-s_0})^{\theta}(D(A^{s_1})) = A^{(s_1-s_0)\theta}(D(A^{s_1})) = D\Big(A^{s_1-(s_1-s_0)\theta}\Big);$$

for $\theta = \frac{1}{2}$ we obtain $[D(A^{s_1}),\, D(A^{s_0})]_{\frac{1}{2}} = D\Big(A^{\frac{s_1+s_0}{2}}\Big)$.

Lemma 1.2.8. *If a sequence (u^n) satisfies $u^n \rightharpoonup u$ in $L^2(0,\, T,\, V)$ and $u^n \to u$ in $L^2(0,\, T,\, H)$, then for any vector function w with components in $C^1(\overline{R} \times [0,\, T])$,*

$$\int_0^T b(u^n(t),\, u^n(t),\, w(t))\, dt \to \int_0^T b(u(t),\, u(t),\, w(t))\, dt.$$

Proof. From

$$b(u^n,\, u^n,\, w) - b(u,\, u,\, w) = b(u^n - u,\, u^n,\, w) + b(u,\, u^n - u,\, w)$$

we obtain

$$\begin{aligned}&\left|\int_0^T b(u^n(t),\, u^n(t),\, w(t)) - b(u(t),\, u(t),\, w(t))\, dt\right| \\ &\leq \int_0^T |b(u^n(t) - u(t),\, u^n(t),\, w(t))|\, dt + \int_0^T |b(u(t),\, u^n(t) - u(t),\, w(t))|\, dt;\end{aligned}$$

[1] Recall that for $-s_0 = s_1 = \frac{1}{2}$ we have $D(A^{1/2}) = V$ and $D(A^{-1/2}) = V'$.

using (1.6) and the skew-symmetry of b in the last two variables, the last sum is bounded by

$$\begin{aligned}
&C_0 \int_0^T \Big[|u^n(t) - u(t)|^{\frac{1}{2}} \|u^n(t) - u(t)\|^{\frac{1}{2}} |u^n(t)|^{\frac{1}{2}} \|u^n(t)\|^{\frac{1}{2}} \|w\| \\
&\qquad + |u(t)|^{\frac{1}{2}} \|u(t)\|^{\frac{1}{2}} |u^n(t) - u(t)|^{\frac{1}{2}} \|u^n(t) - u(t)\|^{\frac{1}{2}} \|w\| \Big] \, dt \\
\leq &C_1 \Big(\int_0^T |u^n(t) - u(t)| \|u^n(t) - u(t)\| \, dt \Big)^{\frac{1}{2}} |u^n(t)|_{L^2(0,\, T,\, V)} \\
&\qquad + C_1 |u(t)|_{L^2(0,\, T,\, V)} \Big(\int_0^T |u^n(t) - u(t)| \|u^n(t) - u(t)\| \, dt \Big)^{\frac{1}{2}};
\end{aligned}$$

but u^n is bounded in $L^2(0,\, T,\, V)$, because $u^n \rightharpoonup u$ in $L^2(0,\, T,\, V)$, then

$$\begin{aligned}
&\Big| \int_0^T b(u^n(t),\, u^n(t),\, w(t)) - b(u(t),\, u(t),\, w(t)) \, dt \Big| \\
\leq &C_2 |u^n(t) - u(t)|^{\frac{1}{2}}_{L^2(0,\, T,\, H)}.
\end{aligned}$$

Hence $\big| \int_0^T b(u^n(t),\, u^n(t),\, w(t)) - b(u(t),\, u(t),\, w(t)) \, dt \big|$ goes to 0 as n goes to $+\infty$. □

1.3 Weak solutions

1.3.1 Existence

The weak existence problem is:

Problem 1.3.1. *Given*

$$F \in L^2(0,\, T,\, V'), \tag{1.9}$$

$$\&$$

$$u_0 \in H, \tag{1.10}$$

$$to\ find$$

$$u \in L^2(0,\, T,\, V), \qquad u_t \in L^1(0,\, T,\, V') \tag{1.11}$$

$$satisfying$$

$$u_t + \nu A u + B u = \nu C u + F \quad on \quad]0,\, T[, \quad ^2 \tag{1.12}$$

$$and$$

$$u(0) = u_0. \tag{1.13}$$

Theorem 1.3.1. *Given F and u_0 satisfying* (1.9) *and* (1.10). *There is at least one function u satisfying* (1.11)-(1.13).

Proof. The proof is analogous to that of theorem 3.1 in [56, ch. 3]; the "new" term $\nu C u$ brings no big problem. Define for each $m \in \mathbb{N}_0$, an approximate solution u^m of (1.12) as:

$$u^m := \sum_{i \leq m} u_i^m(t) W_i, \tag{1.14}$$

[2] To be seen as an equality in V'.

$$((u^m)_t(t),\, W_j) + \nu((u^m(t),\, W_j)) + b(u^m(t),\, u^m(t),\, W_j)$$
$$= \nu(Cu^m(t),\, W_j) + \langle F(t),\, W_j\rangle_{V',V}, \quad t \in [0,\, T], \quad j \le m, \quad (1.15)$$

with
$$u^m(0) = u_0^m. \quad (1.16)$$

Here $\{W_j \mid j \in \mathbb{N}_0\}$ is the system of eigenfunctions of the operator A, $AW_j = k_j W_j$ and u_0^m is the orthogonal projection of u_0 onto $span\{W_i \mid i \le m\}$.

From (1.15) we obtain the nonlinear system of differential equations in the functions $u_i^m, \quad i \le m$:

$$\sum_{i\le m}(u_i^m)_t(W_i,\, W_j) + \nu\sum_{i\le m} u_i^m((W_i,\, W_j)) + \sum_{\substack{i\le m\\ l\le m}} u_i^m u_l^m b(W_i,\, W_l,\, W_j)$$
$$= \nu\sum_{i\le m} u_i^m(CW_i,\, W_j) + \langle F,\, W_j\rangle_{V',V},$$

that reduces to the ODE's system

$$(u_j^m)_t + \nu u_j^m k_j + \sum_{\substack{i\le m\\ l\le m}} u_i^m u_l^m b(W_i,\, W_l,\, W_j)$$
$$= \nu\sum_{i\le m} u_i^m(CW_i,\, W_j) + \langle F,\, W_j\rangle_{V',V}, \qquad j \le m. \quad (1.17)$$

Note that (1.16) is the same as the m scalar conditions

$$u_j^m(0) = \text{the projection of } u_0 \text{ onto } span\{W_j\} = u_{0j}. \quad (1.18)$$

Multiplying (1.15) by u_j^m for each $1 \le j \le m$ and summing up, we find the inequality

$$\frac{d}{dt}|u^m(t)|^2 + 2\nu\|u^m(t)\|^2 \le 2\nu|(Cu^m(t),\, u^m(t))| + 2\langle F(t),\, u^m(t)\rangle_{V',V}$$

or

$$\frac{d}{dt}|u^m(t)|^2 + 2\nu\|u^m(t)\|^2 \le 2\nu K\|u^m(t)\||u^m(t)| + 2\langle F(t),\, u^m(t)\rangle_{V',V}$$
$$\le \nu\|u^m(t)\|^2 + \frac{1}{\nu}K^2\nu^2|u^m(t)|^2 + 2\langle F(t),\, u^m(t)\rangle_{V',V}.$$

Hence

$$\frac{d}{dt}|u^m(t)|^2 + \nu\|u^m(t)\|^2 \le K^2\nu|u^m(t)|^2 + 2\langle F(t),\, u^m(t)\rangle_{V',V}$$
$$\le K^2\nu|u^m(t)|^2 + \frac{2}{\nu}\|F(t)\|_{V'}^2 + \frac{\nu}{2}\|u^m(t)\|^2$$

and
$$\frac{d}{dt}|u^m(t)|^2 + \frac{\nu}{2}\|u^m(t)\|^2 \le K^2\nu|u^m(t)|^2 + \frac{2}{\nu}\|F(t)\|_{V'}^2. \quad (1.19)$$

By the Gronwall inequality, for $s \in [0, T]$

$$|u^m(s)|^2 \le |u_0^m|^2 \exp\Big(\int_0^s K^2\nu\, d\tau\Big) + \int_0^s \frac{2}{\nu}\|F(t)\|_{V'}^2 \exp\Big(-\int_s^t K^2\nu\, d\tau\Big)\, dt$$
$$\le \exp(K^2 T\nu)\Big(|u_0|^2 + \frac{2}{\nu}\int_0^T \|F(t)\|_{V'}^2\, dt\Big). \quad (1.20)$$

Therefore

$$|u^m(s)|^2 \le K_1$$

with K_1 independent of m, i.e.,

$$\textit{the sequence } (u^m) \textit{ remains in a bounded set of } L^\infty(0, T, H). \quad (1.21)$$

From (1.19) and (1.21) we also have

$$\textit{the sequence } (u^m) \textit{ remains in a bounded set of } L^2(0, T, V). \quad (1.22)$$

The rest of the proof follows mainly by classical compactness theorems:

EXTEND u^m TO THE ENTIRE REAL LINE putting

$$\tilde{u}^m(t) := \begin{cases} u^m(t) & \text{if}\, t \in [0, T] \\ 0 & \text{if}\, t \notin [0, T] \end{cases}$$

and denote the Fourier transform of $\tilde{u}^m$ by $\hat{u}^m$.

BOUNDEDNESS IN $\mathcal{H}^\gamma(\mathbb{R}, V, H)$. We must estimate the integral

$$\int_{-\infty}^{+\infty} |\tau|^{2\gamma} |\hat{u}^m(\tau)|^2\, d\tau. \quad (1.23)$$

Equation (1.15) with u^m replaced by $\tilde{u}^m$ results in

$$\frac{d}{dt}(\tilde{u}^m(t), W_j) = \langle \tilde{f}^m, W_j\rangle + (u_0^m, W_j)\delta_0 - (u^m(T), W_j)\delta_T, \quad j \le m \quad (1.24)$$

where δ_0, δ_T are the Dirac distributions at 0 and T and,

$$\begin{aligned} f^m &= F - \nu A u^m - B u^m + \nu C u^m, \\ \tilde{f}^m(t) &:= \begin{cases} f^m(t) & \text{on } [0, T] \\ 0 & \text{outside } [0, T] \end{cases}. \end{aligned}$$

Using the Fourier Transform, (1.24) becomes

$$i\tau(\hat{u}^m(\tau), W_j) = \langle \hat{\tilde{f}}^m(\tau), W_j\rangle + (u_0^m, W_j)(2\pi)^{-1/2} - (u^m(T), W_j)(2\pi)^{-1/2}\exp(-iT\tau),$$

and multiplying by $\hat{u}_i^m(\tau)$ and adding the obtained m equations we arrive to

$$i\tau|\hat{u}^m(\tau)|^2 = \langle \hat{f}^m(\tau), \hat{u}^m(\tau)\rangle + (2\pi)^{-1/2}\Big[(u_0^m, \hat{u}^m(\tau)) - (u^m(T), \hat{u}^m(\tau))\exp(-iT\tau)\Big]. \quad (1.25)$$

From

$$\|f^m(t)\|_{V'} \le \|F(t)\|_{V'} + D\|u^m(t)\| + D\|u^m(t)\|^2,$$

(1.9) and (1.22), we have that the integral

$$\int_0^T \|f^m(t)\|_{V'}\,dt \le \int_0^T \|F(t)\|_{V'} + D\|u^m(t)\| + D\|u^m(t)\|^2\,dt$$

remains bounded. Hence [3]

$$\sup_{\tau\in\mathbb{R}} \|\hat{f}^m(\tau)\|_{V'} \le \text{ const}, \quad \forall m \in \mathbb{N}_0.$$

By (1.20) both $|u^m(0)|$ and $|u^m(T)|$ are finite so, by (1.25),

$$|\tau||\hat{u}^m(\tau)|^2 \le C_1\|\hat{u}^m(\tau)\| + C_2|\hat{u}^m(\tau)| \le D\|\hat{u}^m(\tau)\| \tag{1.26}$$

for suitable constants C_1, C_2 and D.

Fix $\gamma < \frac{1}{4}$ and define the real function $Q(x) := \frac{x^{2\gamma}+x}{1+x}, \quad x \in [0, +\infty[$. Q is continuous and bounded,[4] then we can find a constant $D_1 \in \mathbb{R}^+$ such that for all $\tau \in \mathbb{R}$, $Q(|\tau|) \le D_1$; from which we derive

$$|\tau|^{2\gamma} \le D_1 \frac{1+|\tau|}{1+|\tau|^{1-2\gamma}}.$$

Therefore the integral (1.23) is bounded by

$$\begin{aligned} &D_1 \int_{-\infty}^{+\infty} \frac{1+|\tau|}{1+|\tau|^{1-2\gamma}} |\hat{u}^m(\tau)|^2\,d\tau \\ \le &D_2 \int_{-\infty}^{+\infty} \frac{\|\hat{u}^m(\tau)\|}{1+|\tau|^{1-2\gamma}}\,d\tau + D_3 \int_{-\infty}^{+\infty} \|\hat{u}^m(\tau)\|^2\,d\tau, \quad \text{[using (1.26)]}. \end{aligned}$$

By the Parseval Equality and (1.22), there is a constant D_4 such that

$$D_3 \int_{-\infty}^{+\infty} \|\hat{u}^m(\tau)\|^2\,d\tau \le D_4, \qquad \forall m \tag{1.27}$$

For the integral $D_2 \int_{-\infty}^{+\infty} \frac{\|\hat{u}^m(\tau)\|}{1+|\tau|^{1-2\gamma}}\,d\tau$ we apply Schwartz Inequality and Parseval Equality to obtain

$$D_2 \int_{-\infty}^{+\infty} \frac{\|\hat{u}^m(\tau)\|}{1+|\tau|^{1-2\gamma}}\,d\tau \le D_2 \left(\int_{-\infty}^{+\infty} \frac{1}{(1+|\tau|^{1-2\gamma})^2}\,d\tau\right)^{\frac{1}{2}} \left(\int_0^T \|u^m(t)\|^2\,dt\right)^{\frac{1}{2}} \quad [5]$$

and, this product is finite and bounded as $m \to +\infty$, i.e., there is a constant D_5 such that

$$D_2 \int_{-\infty}^{+\infty} \frac{\|\hat{u}^m(\tau)\|}{1+|\tau|^{1-2\gamma}}\,d\tau \le D_5, \qquad \forall m. \tag{1.28}$$

[3] It is known that $|\hat{f}(\tau)| \le C\int_{\mathbb{R}} |f(t)|\,dt$. See for example [53].

[4] $\lim_{x\to+\infty} Q(x) = 1$, $Q(0) = 0$

[5] Since $\gamma < \frac{1}{4}$, $\frac{1}{1+|\tau|^{1-2\gamma}} \in L^2(\mathbb{R})$. Indeed $\int_{\mathbb{R}} \frac{1}{(1+|\tau|^{1-2\gamma})^2}\,d\tau = 2\int_0^{+\infty} \frac{1}{(1+\tau^{1-2\gamma})^2}\,d\tau$ and, putting $x = 1+\tau^{1-2\gamma}$ we see that the last integral equals $2\int_1^{+\infty} \frac{1}{x^2}\frac{1}{1-2\gamma}(x-1)^{\frac{2\gamma}{1-2\gamma}}\,dx \le \frac{2}{1-2\gamma}\int_1^{+\infty} \frac{1}{x^{2-\frac{2\gamma}{1-2\gamma}}}\,dx$ and the latter converges if $2 - \frac{2\gamma}{1-2\gamma} > 1 \leftrightarrow \gamma < \frac{1}{4}$.

By (1.27) and (1.28) we conclude that the integral (1.23) is finite:

$$\int_{-\infty}^{+\infty} |\tau|^{2\gamma} |\hat{u}^m(\tau)|^2 \, d\tau \leq D_4 + D_5 =: E.$$

The finiteness of (1.23) with (1.22) implies that

$$\textit{the sequence } (u^m) \textit{ remains in a bounded set of } \mathcal{H}^\gamma(\mathbb{R}, V, H). \tag{1.29}$$

The limit. From lemma 1.2.6 and (1.21) there exists of a subsequence $(u^{\sigma(m)})$ of u^m and $u \in L^\infty(0, T, H)$ such that

$$u^{\sigma(m)} \rightharpoonup_* u, \text{ in } L^\infty(0, T, H). \quad ^6$$

Analogously, lemma 1.2.5 and (1.22) implies the existence of a subsequence $(u^{\alpha(\sigma(m))})$ of $u^\sigma(m)$ and $v \in L^2(0, T, V)$ such that

$$u^{\alpha(\sigma(m))} \rightharpoonup v, \text{ in } L^2(0, T, V).$$

The sequence (u^m) is in the space $\mathcal{H}^\gamma_{[0,T]}(\mathbb{R}, V, H)$ which injection in $L^2(\mathbb{R}, H)$ is compact due to theorem 1.2.4.[7] Then (1.29) implies the existence of a subsequence $\beta(\alpha(\sigma(m)))$ of $\alpha(\sigma(m))$ and $w \in L^2(0, T, H)$ satisfying

$$u^{\beta(\alpha(\sigma(m)))} \rightarrow w, \text{ in } L^2(0, T, H).$$

We put $\eta := \beta \circ \alpha \circ \sigma$ and we obtain

$$u = v = w \in L^2(0, T, H) \cap L^2(0, T, V) \cap L^\infty(0, T, H)$$

and

$$u^{\eta(m)} \rightharpoonup_* u, \text{ in } L^\infty(0, T, H); \tag{1.30}$$

$$u^{\eta(m)} \rightharpoonup u, \text{ in } L^2(0, T, V); \tag{1.31}$$

$$u^{\eta(m)} \rightarrow u, \text{ in } L^2(0, T, H). \tag{1.32}$$

Indeed from

$$\int_0^T (u^{\eta(m)} - u) f_1 \, dt \rightarrow 0 \quad \forall f_1 \in L^1(0, T, H)$$

$$\int_0^T (u^{\eta(m)} - v) f_2 \, dt \rightarrow 0 \quad \forall f_2 \in L^2(0, T, V')$$

$$\int_0^T (u^{\eta(m)} - w) f_3 \, dt \rightarrow 0 \quad \forall f_3 \in L^2(0, T, H)$$

and from the inclusion $L^2(0, T, H) \subset L^1(0, T, H) \cap L^2(0, T, V')$ we conclude that, since both u, v and w are in $L^2(0, T, H)$, then both u, v and w coincide with the limit of $u^{\eta(m)}$ in $L^2(0, T, H)$.

[6] If wanted, without lack of generality we may assume $u^m \rightharpoonup_* u$, in $L^\infty(0, T, H)$.

[7] With $V = X_0$, & $H = X = X_1$.

ENDING. Multiplying (1.15) by a function $\phi \in C^1([0,\, T])$ and, integrating we obtain:

$$\begin{aligned}
&\int_0^T ((u^m)'(t),\, W_j\phi(t))\, dt + \int_0^T \nu((u^m(t),\, W_j\phi(t)))\, dt \\
&+ \int_0^T b(u^m(t),\, u^m(t),\, W_j\phi(t))\, dt = \int_0^T \nu(Cu^m(t),\, W_j\phi(t))\, dt + \int_0^T \langle F(t),\, W_j\phi(t)\rangle\, dt \\
\iff &- \int_0^T (u^m(t),\, W_j\phi'(t))\, dt + \nu \int_0^T ((u^m(t),\, W_j\phi(t)))\, dt \\
&+ \int_0^T b(u^m(t),\, u^m(t),\, W_j\phi(t))\, dt = (u_0^m,\, W_j)\phi(0) - (u^m(T),\, W_j)\phi(T) \\
&+ \int_0^T \nu(Cu^m(t),\, W_j\phi(t))\, dt + \int_0^T \langle F(t),\, W_j\phi(t)\rangle\, dt.
\end{aligned}$$

Due to lemma 1.2.8 we can take the limit in the nonlinear term and, by (1.31) we can take the limit in the linear terms. Hence

$$\begin{aligned}
&- \int_0^T (u(t),\, W_j\phi'(t))\, dt + \nu \int_0^T ((u(t),\, W_j\phi(t)))\, dt + \int_0^T b(u(t),\, u(t),\, W_j\phi(t))\, dt \\
&= \nu \int_0^T (Cu(t),\, W_j\phi(t))\, dt + (u_0,\, W_j)\phi(0) - (u(T),\, W_j)\phi(T) + \int_0^T \langle F(t),\, W_j\phi(t)\rangle\, dt \quad (1.33)
\end{aligned}$$

This equation, being true for all W_j, by linearity is true for any finite combination of functions in $\mathcal{W}$ and, by continuity, will be true for all $v \in V$. Then taking a test function $\phi \in \mathcal{D}(]0,\, T[)$ in (1.33) we conclude that

$$\begin{aligned}
&- \int_0^T (u(t),\, v\phi'(t))\, dt + \nu \int_0^T ((u(t),\, v\phi(t)))\, dt + \int_0^T b(u(t),\, u(t),\, v\phi(t))\, dt \\
&=\nu \int_0^T (Cu(t),\, v\phi(t))\, dt + (u_0,\, v)\phi(0) - (u(T),\, v)\phi(T) + \int_0^T \langle F(t),\, v\phi(t)\rangle\, dt \qquad (1.34)
\end{aligned}$$

and then

$$\begin{aligned}
&- \int_0^T (u(t),\, v\phi'(t))\, dt + \nu \int_0^T ((u(t),\, v\phi(t)))\, dt + \int_0^T b(u(t),\, u(t),\, v\phi(t))\, dt \\
&=\nu \int_0^T (Cu(t),\, v\phi(t))\, dt + \int_0^T \langle F(t),\, v\phi(t)\rangle\, dt \qquad (1.35)
\end{aligned}$$

what means that equation (1.12) is satisfied in the distribution sense and then, by lemma 1.2.3, $-\nu Au - Bu + \nu Cu + F$ is a primitive for u, i.e., (1.12) is satisfied as an equality in V'. Moreover u coincides a.e. with a continuous function from $[0,\, T]$ into V'.

Multiplying (1.12) by ϕv with $\phi \in C^1([0,\, T])$, such that $\phi(0) = 1$, $\phi(T) = 0$, and $v \in V$, we obtain

$$\begin{aligned}
&- \int_0^T (u(t),\, v\phi'(t))\, dt + \nu \int_0^T ((u(t),\, v\phi(t)))\, dt + \int_0^T b(u(t),\, u(t),\, v\phi(t))\, dt \\
&=\nu \int_0^T (Cu(t),\, v\phi(t))\, dt + (u(0),\, v) + \int_0^T \langle F(t),\, v\phi(t)\rangle\, dt. \qquad (1.36)
\end{aligned}$$

The equation resulting from (1.34) with the same ϕ as in (1.36) is

$$\begin{aligned}
&-\int_0^T (u(t),\, v\phi'(t))\,dt + \nu\int_0^T ((u(t),\, v\phi(t)))\,dt + \int_0^T b(u(t),\, u(t),\, v\phi(t))\,dt \\
=&\nu\int_0^T (Cu(t),\, v\phi(t))\,dt + (u_0,\, v) + \int_0^T \langle F(t),\, v\phi(t)\rangle\,dt
\end{aligned} \tag{1.37}$$

From (1.36) and (1.37) we conclude that (1.13) is satisfied (and, this finishes the proof of theorem 1.3.1). □

1.3.2 Uniqueness

It turns out, for we may see again [56], that

Theorem 1.3.2. *The solution of problem 1.3.1 given by theorem 1.3.1 is unique. Moreover it is a.e. equal to a continuous function from* $[0,\, T]$ *into* H *and,*

$$u(t) \to u(t_1), \quad \textit{in } H \quad \textit{as } t \to t_1, \quad t_1 \in [0,\, T]. \tag{1.38}$$

In particular

$$u(t) \to u_0, \quad \textit{in } H \quad \textit{as } t \to 0; \qquad u(t) \to u(T), \quad \textit{in } H \quad \textit{as } t \to T.$$

Proof. Consider two solutions u, v of the problem and put $w := u - v$. Then

$$\begin{aligned}
w' =&u' - v' = \tilde{F} - \nu Au - Bu + \nu Cu - (\tilde{F} - \nu Av - Bv + \nu Cu) \\
=&-\nu Aw - Bu + Bv + \nu Cw \\
\text{and}& \\
w(0) =&0;
\end{aligned}$$

Taking the scalar product with w in the duality between V and V' we obtain

$$\langle w',\, w\rangle + \nu\langle Aw,\, w\rangle = \langle Bv,\, w\rangle - \langle Bu,\, w\rangle + \nu(Cw,\, w)$$

and

$$\begin{aligned}
\frac{d}{dt}|w(t)|^2 + 2\nu\|w(t)\|^2 &= 2b(v(t),\, v(t),\, w(t)) - 2b(u(t),\, u(t),\, w(t)) + \nu(Cw,\, w) \\
&\le 2(C|w(t)|\|w(t)\|\|v(t)\|) + \nu K\|w\||w|.
\end{aligned}$$

By suitable Young inequalities and by the Gronwall inequality, we can arrive to

$$|w(t)|^2 \le D_1|w(0)|^2 = 0.$$

That $u \in C([0,\, T],\, H)$ follows from lemma 1.2.7. Note that $|u_t|_{V'} \le C_1(\|u\| + |u|\|u\| + |F|_{V'})$, in particular $u_t \in L^2(0,\, T,\, V')$. □

1.3.3 Continuity on the initial data

Theorem 1.3.3. *The map*

$$\begin{aligned}\mathbb{S} : H \times L^2(0,\, T,\, V') \times]0,\, +\infty[&\to C([0,\, T],\, H)\\ (u_0,\, F,\, \nu) &\mapsto u\end{aligned}$$

is continuous. Here $u \in C([0,\, T],\, H)$ is the unique solution of problem 1.3.1.

Theorem 1.3.4. *The map*

$$\begin{aligned}\mathbb{S}_2 : H \times L^2(0,\, T,\, V') \times]0,\, +\infty[&\to L^2(0,\, T,\, V)\\ (u_0,\, F,\, \nu) &\mapsto u\end{aligned}$$

is continuous. Here $u \in L^2(0,\, T,\, V)$ is the unique solution of problem 1.3.1.

The proofs of theorems 1.3.3 and 1.3.4 follow mainly by playing with Young and Gronwall inequalities and with the estimates (1.2)–(1.6) for the bilinear term. Details and suggestions may be found in [48, 45].

1.4 Strong solutions

Asking for some more regularity in the initial condition and external force and proceeding as in the weak case, we obtain more regularity for the solution:

Problem 1.4.1. *Given*

$$F \in L^2(0,\, T,\, H), \tag{1.39}$$

&

$$u_0 \in V, \tag{1.40}$$

to find

$$u \in L^2(0,\, T,\, D(A)) \cap L^\infty(0,\, T,\, V), \qquad u_t \in L^2(0,\, T,\, H) \tag{1.41}$$

satisfying

$$u_t + \nu A u + B u = \nu C u + F \quad \text{on} \quad]0,\, T[, \quad ^{8} \tag{1.42}$$

and

$$u(0) = u_0. \tag{1.43}$$

1.4.1 Existence

Theorem 1.4.1. *Given F and u_0 satisfying* (1.39) *and* (1.40)*. There is at least one function u satisfying* (1.41)-(1.43)*.*

1.4.2 Uniqueness

Theorem 1.4.2. *The solution of problem 1.4.1 is unique. Moreover it is a.e. equal to a continuous function from $[0,\, T]$ into V.*

[8] Like in (1.12), to be seen as an equality in V'.

1.4.3 Continuity on the initial data

Theorem 1.4.3. *The map*

$$\begin{aligned}\mathbb{S}_s : V \times L^2(0,\,T,\,H)\times]0,\,+\infty[&\to C([0,\,T],\,V)\\ (u_0,\,F,\,\nu) &\mapsto u\end{aligned}$$

is continuous. Where $u \in C([0,\,T],\,V)$ is the unique solution of problem 1.4.1.

Theorem 1.4.4. *The map*

$$\begin{aligned}\mathbb{S}_{2s} : V \times L^2(0,\,T,\,H)\times]0,\,+\infty[&\to L^2(0,\,T,\,D(A))\\ (u_0,\,F,\,\nu) &\mapsto u\end{aligned}$$

is continuous. Where $u \in L^2(0,\,T,\,D(A))$ is the unique solution of problem 1.4.1.

1.4.4 The $L^2(0,\,T,\,H)$-norm of u_t

Multiplying

$$u_t = -\nu Au - Bu + \nu Cu + F$$

by u_t, we obtain

$$\begin{aligned}|u_t|^2 &= -\frac{1}{2}\nu\frac{d}{dt}\|u\|^2 - (Bu,\,u_t) + \nu(Cu,\,u_t) + (F,\,u_t)\\ &\le -\frac{1}{2}\nu\frac{d}{dt}\|u\|^2 + D\|u\||u|_{[2]}|u_t| + D\|u\||u_t| + |F||u_t|;\end{aligned}$$

so

$$\frac{1}{2}|u_t|^2 \le -\frac{1}{2}\nu\frac{d}{dt}\|u\|^2 + D_1\|u\|^2|u|^2_{[2]} + D_1\|u\|^2 + D_1|F|^2$$

and

$$|u_t|^2_{L^2(0,\,T,\,H)} \le D_2\left[|u|^2_{C(0,\,T,\,V)}\Big(1 + |u|^2_{L^2(0,\,T,\,D(A))}\Big) + |F|^2_{L^2(0,\,T,\,H)}\right].$$

Therefore the norm of u_t is somehow bounded by the norms of u and F.

Moreover fixing $(u_0,\,F,\,\nu)$ in the product space $V \times L^2(0,\,T,\,H) \times \mathbb{R}^+$ and considering another element $(v_0,\,G,\,\eta)$ close to $(u_0,\,F,\,\nu)$, for the derivative w_t of the difference $w = u - v$ we find

$$w_t = -\nu Aw + (\eta-\nu)Av + Bv - Bu + \nu Cw + (\nu-\eta)Cv + F - G;$$

multiplying by w_t we obtain

$$\begin{aligned}|w_t|^2 \le& -\frac{1}{2}\nu\frac{d}{dt}\|w\|^2 + |\eta-\nu|\Big(|v|_{[2]} + K\|v\|\Big)|w_t| + \nu K\|w\||w_t|\\ &+ |F-G||w_t| + K\|w\|\Big(|u|_{[2]} + |v|_{[2]}\Big)|w_t| \quad [9]\end{aligned}$$

and the estimate

$$\begin{aligned}|w_t|^2_{L^2(0,\,T,\,H)} \le D\Bigg[&|w|^2_{C(0,\,T,\,V)}\Big(1 + |u|^2_{L^2(0,\,T,\,D(A))} + |v|^2_{L^2(0,\,T,\,D(A))}\Big)\\ &+ |F-G|^2_{L^2(0,\,T,\,H)} + |\eta-\nu|^2|v|^2_{L^2(0,\,T,\,D(A))}\Bigg].\end{aligned}$$

[9] The product $(Bv - Bu,\,w_t)$ is equal to $b(v-u,\,u,\,w_t) + b(v,\,v-u,\,w_t)$.

Since $(v_0,\, G,\, \eta)$ is close to $(u_0,\, F,\, \nu)$, by theorem 1.4.4, we have that $|v|^2_{L^2(0,\,T,\,D(A))}$ is close to $|u|^2_{L^2(0,\,T,\,D(A))}$ and we may write

$$|w_t|^2_{L^2(0,\,T,\,H)} \le D_1 \Big[|w|^2_{C(0,\,T,\,V)} \Big(1 + |u|^2_{L^2(0,\,T,\,D(A))}\Big) \\ + |F - G|^2_{L^2(0,\,T,\,H)} + |\eta - \nu|^2 \Big(1 + |u|^2_{L^2(0,\,T,\,D(A))}\Big)\Big].$$

Therefore the derivative u_t varies continuously in $L^2(0,\,T,\,H)$ when the data varies in the product $V \times L^2(0,\,T,\,H) \times \mathbb{R}^+$.

1.5 Change of variables. Relaxation

For the controlled version with $F + v$ in the place of F we have existence, uniqueness and continuity in the data $(u_0,\, F + v,\, \nu)$. F is a fixed external force, with the properties of F, and v will be our control we suppose to be an essentially bounded function taking values on H and so, $F + v$ belong to the same set F does: $(L^2(0,\,T,\,V')$ or $L^2(0,\,T,\,H))$. If we consider the initial data as $(u_0,\, F,\, v,\, \nu)$, by theorems 1.3.1, 1.3.2, 1.3.3 and 1.3.4 we derive some corollaries on existence, uniqueness and continuity of weak solutions corresponding to the data $(u_0,\, F,\, v,\, \nu) \in H \times L^2(0,\,T,\,V') \times L^\infty(0,\,T,\,H) \times]0,\,+\infty[$. Analogously for strong solutions corresponding to data $(u_0,\, F,\, v,\, \nu) \in V \times L^2(0,\,T,\,H) \times L^\infty(0,\,T,\,H) \times]0,\,+\infty[$.

1.5.1 Change of variables: $u \mapsto y$

If we make the change of variables

$$u = y + \mathbb{I}v$$

where $\mathbb{I}$ is the primitive operator — $[\mathbb{I}v](t) = \int_0^t v(\tau)\, d\tau$, from

$$u_t = -\nu A u - B u + \nu C u + F + v$$

we arrive to the equation

$$y_t = -\nu A(y + \mathbb{I}v) - B(y + \mathbb{I}v) + \nu C(y + \mathbb{I}v) + F.$$

Note that the function v appears only implicitly in the last equation. Now we forget that $\mathbb{I}v$ is a primitive of an essentially bounded function and replace it by P in the equation. Since v is a low modes forcing it takes value in a finite-dimensional space $\mathbb{F} \subset H$ and, $\mathbb{I}v$ being a primitive we have $\mathbb{I}v \in C([0,\,T],\,H)$. First we restrict ourselves to the case $\mathbb{F} \subset D(A)$ but, on the other side we take P in the larger space $L^4(0,\,T,\,D(A))$ instead of $C([0,\,T],\,D(A))$.

1.5.2 Weak case

Similarly as we have done in [48] we consider the weak "Y-problem":

Problem 1.5.1. *Given*

$$F \in L^2(0, T, V'),, \quad P \in L^4(0, T, D(A)) \tag{1.44}$$

&

$$y_0 \in H, \tag{1.45}$$

to find

$$y \in L^2(0, T, V), \qquad y_t \in L^1(0, T, V') \tag{1.46}$$

satisfying

$$y_t + \nu A(y + P) + B(y + P) = \nu C(y + P) + F \quad \text{on} \quad]0, T[, \tag{1.47}$$

and

$$y(0) = y_0. \tag{1.48}$$

Existence

We have the theorem:

Theorem 1.5.1. *Given F, P and y_0 satisfying* (1.44) *and* (1.45). *There is at least one function y satisfying* (1.46)-(1.48).

Proof. Like in the proof of theorem 1.3.1 we start by defining an approximate solution

$$y^m = \sum_{i \le m} y_i^m(t) W_i$$

for each $m \in \mathbb{N}_0$ and arrive to the equation

$$\begin{aligned}((y^m)_t, y^m) + \nu(A(y^m + P^m), y^m) &+ (B(y^m + P^m), y^m) \\ &= \nu(C(y^m + P^m), y^m) + \langle F, y^m \rangle_{V',V}. \quad ^{10}\end{aligned} \tag{1.49}$$

From which we derive

$$\begin{aligned}&\frac{d}{dt}|y^m|^2 + 2\nu\|y^m\|^2 \\ =& -2\nu((P^m, y^m)) - 2b(y^m, P^m, y^m) \\ &+ 2b(P^m, y^m, P^m) + 2\nu(C(y^m + P^m), y^m) + 2\langle F, y^m \rangle_{V',V}\end{aligned}$$

and, playing again with the estimates for the bilinear operator we may conclude that the sequence (u^m) in a bounded set of $L^\infty(0, T, H) \cap L^2(0, T, V)$ and then proceed as in the proof of theorem 1.3.1. □

Uniqueness

Theorem 1.5.2. *The solution of problem 1.5.1 given by theorem 1.5.1 is unique. Moreover it is a.e. equal to a continuous function from $[0, T]$ into H.*

[10] Where P^m is the projection of P onto $span\{W_i \mid i \le m\}$.

Continuity

Theorem 1.5.3. *The map*

$$\mathbb{Y} : H \times L^2(0, T, V') \times L^4(0, T, D(A)) \times]0, +\infty[\to C([0, T], H)$$
$$(y_0, F, P, \nu) \mapsto y$$

is continuous. Where y is the unique solution of problem (1.5.1) *corresponding to the data* (y_0, F, P, ν).

Theorem 1.5.4. *The map*

$$\mathbb{Y}_2 : H \times L^2(0, T, V') \times L^4(0, T, D(A)) \times]0, +\infty[\to L^2(0, T, V)$$
$$(y_0, F, P, \nu) \mapsto y$$

is continuous. Where y is the unique solution of problem (1.5.1) *corresponding to the data* (y_0, F, P, ν).

1.5.3 Strong case

Consider the strong Y-problem:

Problem 1.5.2. *Given*

$$F \in L^2(0, T, H),, \quad P \in L^4(0, T, D(A)) \tag{1.50}$$

&

$$y_0 \in V, \tag{1.51}$$

to find

$$y \in L^2(0, T, D(A)) \cap L^\infty(0, T, V), \quad y_t \in L^2(0, T, H) \tag{1.52}$$

satisfying

$$y_t + \nu A(y + P) + B(y + P) = \nu C(y + P) + F \quad \text{on} \quad]0, T[, \tag{1.53}$$

and

$$y(0) = y_0. \tag{1.54}$$

Existence

Theorem 1.5.5. *Given F, P and u_0 satisfying* (1.50) *and* (1.51). *There is at least one function y satisfying* (1.52)-(1.54).

Uniqueness

Theorem 1.5.6. *The solution of problem 1.5.2 given by theorem 1.5.5 is unique. Moreover it is a.e. equal to a continuous function from* $[0, T]$ *into* V.

Continuity

Theorem 1.5.7. *The map*

$$\begin{aligned}\mathbb{Y}_s : V \times L^2(0,\,T,\,H) \times L^4(0,\,T,\,D(A)) \times]0,\,+\infty[&\to C([0,\,T],\,V)\\ (y_0,\,F,\,P,\,\nu) &\mapsto y\end{aligned}$$

is continuous. Where y is the unique solution of problem (1.5.2) *corresponding to the data* $(y_0,\,F,\,P,\,\nu)$.

Theorem 1.5.8. *The map*

$$\begin{aligned}\mathbb{Y}_{2s} : V \times L^2(0,\,T,\,H) \times L^4(0,\,T,\,D(A)) \times]0,\,+\infty[&\to L^2([0,\,T],\,D(A))\\ (y_0,\,F,\,P,\,\nu) &\mapsto y\end{aligned}$$

is continuous. Where y is the unique solution of problem (1.5.2) *corresponding to the data* $(y_0,\,F,\,P,\,\nu)$.

1.5.4 The $L^2(0,\,T,\,H)$-norm of y_t

Multiplying

$$y_t = -\nu A(y+P) - B(y+P) + \nu C(y+P) + F$$

by y_t, we obtain

$$\begin{aligned}|y_t|^2 =& -\frac{1}{2}\nu\frac{d}{dt}\|y\|^2 + \nu|P|_{[2]}|y_t| + |(B(y+P),\,y_t)| + K\|y+P\||y_t| + |F||y_t|\\ \le& -\frac{1}{2}\nu\frac{d}{dt}\|y\|^2 + \nu|P|_{[2]}|y_t|\\ &+ D\Big(\|y\||y|_{[2]} + \|y\||P|_{[2]} + \|P\||P|_{[2]}\Big)|y_t| + K\|y+P\||y_t| + |F||y_t|;\end{aligned}$$

so

$$\frac{1}{2}|y_t|^2 \le -\frac{1}{2}\nu\frac{d}{dt}\|y\|^2 + D_1\|y\|^2\Big(1 + |y|^2_{[2]} + |P|^2_{[2]}\Big) + D_1|P|^2_{[2]}(1+\|P\|^2) + |F|^2;$$

and

$$\begin{aligned}|y_t|^2_{L^2(0,\,T,\,H)} \le D_2\Bigg[&|y|^2_{C(0,\,T,\,V)}\Big(1 + |y|^2_{L^2(0,\,T,\,D(A))} + |P|^2_{L^4(0,\,T,\,D(A))}\Big)\\ &+ |P|^2_{L^4(0,\,T,\,D(A))} + |P|^4_{L^4(0,\,T,\,D(A))} + |F|^2_{L^2(0,\,T,\,H)}\Bigg].\end{aligned}$$

Therefore the norm of the derivative y_t is somehow bounded by the norms of y, P and F.

Moreover fixing $(y_0,\,F,\,P,\,\nu)$ in the product space $V \times L^2(0,\,T,\,H) \times L^4(0,\,T,\,D(A)) \times \mathbb{R}^+$ and considering another element $(z_0,\,G,\,Q,\,\eta)$ close to $(y_0,\,F,\,P,\,\nu)$, for the derivative w_t of the difference $w = y - z$ we find the estimate

$$\begin{aligned}|w_t|^2 \le& -\frac{1}{2}\nu\frac{d}{dt}\|w\|^2 + \nu|P-Q|_{[2]}|w_t| + |\eta-\nu||z+Q|_{[2]}|w_t|\\ &+ K\|w\||w_t| + K\|P-Q\||w_t| + K|\eta-\nu|\|z+Q\||w_t| + |F-G||w_t|\\ &+ |(B(z+Q) - B(y+P),\,w_t)|;\end{aligned}$$

expanding the last term as $B(a+b) = Ba+Bb+B(a, b)+B(b, a)$ and using $B(a, b)-B(c, d) = B(a-c, b) + B(c, b-d)$ we arrive to

$$\begin{aligned}
|(Bz - By, w_t)| &\le K\|w\|(|z|_{[2]} + |y|_{[2]})|w_t| \\
|(BQ - BP, w_t)| &\le K\|P - Q\|(|P|_{[2]} + |Q|_{[2]})|w_t| \\
|(B(z, Q) - B(y, P), w_t)| &\le K(\|w\||Q|_{[2]} + |P - Q|_{[2]}\|y\|)|w_t| \\
|(B(Q, z) - B(P, y), w_t)| &\le K(\|w\||P|_{[2]} + |P - Q|_{[2]}\|z\|)|w_t|.
\end{aligned}$$

Thus

$$\begin{aligned}
|(B(z+Q) - B(y+P), w_t)| \le &K\|w\|(|P|_{[2]} + |Q|_{[2]} + |y|_{[2]} + |z|_{[2]}) \\
&+ K|P - Q|_{[2]}(|P|_{[2]} + |Q|_{[2]} + \|y\| + \|z\|)
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{2}|w_t|^2 \le &-\frac{1}{2}\nu\frac{d}{dt}\|w\|^2 \\
&+ D\|w\|^2\Big(1 + |P|_{[2]}^2 + |Q|_{[2]}^2 + |y|_{[2]}^2 + |z|_{[2]}^2\Big) + D|\eta - \nu|^2\Big(|Q|_{[2]}^2 + |z|_{[2]}^2\Big) \\
&+ D|P - Q|_{[2]}^2\Big(1 + |P|_{[2]}^2 + |Q|_{[2]}^2 + \|y\|^2 + \|z\|^2\Big) + D|F - G|^2.
\end{aligned}$$

Since (z_0, G, Q, η) is close to (y_0, F, P, ν), by the continuity results on the initial data, we may arrive to the estimate

$$|w_t|_{L^2(0, T, H)}^2 \le D_1\Big(|w|_{C([0, T], V)}^2 + |P - Q|_{L^4(0, T, D(A))}^2 + |\eta - \nu|^2 + |F - G|_{L^2(0, T, H)}^2\Big).$$

Therefore y_t varies continuously in $L^2(0, T, H)$ when the data varies in the product $V \times L^2(0, T, H) \times L^4(0, T, D(A)) \times \mathbb{R}^+$.

1.5.5 Continuity in relaxation metric

We begin with a definition:

Definition 1.5.1. *Given a finite dimensional normed space $\mathbb{F} \subset H$ and a basis $\beta = \{e_i \mid i = 1, \ldots, p\}$ for $\mathbb{F}$; the β-***relaxation metric** *in $L^1(0, T, \mathbb{F})$ is defined by the norm*

$$|\tilde{g}|_{rx} = |\tilde{g}|_{rx(\beta)} := \max_{t_1, t_2 \in [0, T]}\left|\int_{t_1}^{t_2} g(\tau)\, d\tau\right|_{l_1}; \quad g = (g_1, \ldots, g_p),\ \tilde{g} = \sum_{i=1}^{p} g_i e_i. \qquad [11] \qquad (1.55)$$

Consider, also, the β-**w-relaxation metric** on $L^1(0, T, \mathbb{F})$ defined by the norm

$$|\tilde{g}|_{wrx} = |\tilde{g}|_{wrx(\beta)} := \max_{t \in [0, T]}\left|\int_0^t g(\tau)\, d\tau\right|_{l_1}. \qquad (1.56)$$

[11] Recall that for $x = (x_1, \ldots, x_p) \in \mathbb{R}^p$, $|x|_{l_1} := \sum_{i=1}^p |x_i|$.

Remark 1.5.1. *It is easy to see that the identity map*

$$\Big(L^1(0,\,T,\,\mathbb{F}),\,|\cdot|_{rx}\Big) \to \Big(L^1(0,\,T,\,\mathbb{F}),\,|\cdot|_{wrx}\Big)$$

and the map

$$\begin{aligned}\mathbb{I} : L^\infty_{wrx}([0,\,T],\,\mathbb{F}) &\to C([0,\,T],\,\mathbb{F})\\ v &\mapsto \mathbb{I}v\end{aligned}$$

are continuous. Where the subscript "wrx" means that we are considering w-relaxation metric on the set $L^\infty([0,T],\,\mathbb{F})$. Since all the norms in $\mathbb{F}$ are equivalent, in the last space $C([0,\,T],\,\mathbb{F})$ we may consider in $\mathbb{F}$ any norm.

Recall that by definition, the map $\mathbb{S}$ (see theorem 1.3.3) gives us the weak solution, in $C([0,\,T],\,H)$, of the equation for an initial data in $\Pi := H\times L^2(0,\,T,\,V')\times L^\infty([0,\,T],\mathbb{F})\times\mathbb{R}^+$. Changing the topology on the third factor of the previous product to the w-relaxation one, we arrive to the space $L^\infty_{wrx}([0,\,T],\,\mathbb{F})$ and we define the function $\mathbb{S}_{wrx}$ as the function defined in the product $\Pi_{wrx} := H\times L^2(0,\,T,\,V')\times L^\infty_{wrx}([0,\,T],\mathbb{F})\times\mathbb{R}^+$ and taking the same values as $\mathbb{S}$.

$\diamond$ **The case $\mathbb{F}\subset D(A)$:**

Proposition 1.5.9. *The map $\mathbb{S}_{wrx}$ is continuous.*

Proof. Put $\mathbb{I}_\circ(u_0,\,F,\,v,\,\nu) := \mathbb{I}v$. By remark 1.5.1 and theorem 1.5.3 the map

$$\begin{aligned}\mathbb{Y}_{wrx} : \Pi_{wrx} &\to C([0,\,T],\,H)\\ (u_0,\,F,\,v,\,\nu) &\mapsto \mathbb{Y}(u_0,\,F,\,\mathbb{I}v,\,\nu) = \mathbb{Y}\circ\mathbb{I}^\circ(u_0,\,F,\,v,\,\nu)\end{aligned}$$

is continuous; where $\mathbb{I}^\circ(u_0,\,F,\,v,\,\nu)$ stays for $(u_0,\,F,\,\mathbb{I}v,\,\nu)$.

By the equality $\mathbb{S}_{wrx} = \mathbb{Y}_{wrx} + \mathbb{I}_\circ$ we conclude the continuity of $\mathbb{S}_{wrx}$. $\square$

Analogously, by remark 1.5.1 and theorems 1.5.4, 1.5.7 and 1.5.8, we can prove the continuity on relaxation metric of the maps $\mathbb{S}_2$, $\mathbb{S}_s$ and $\mathbb{S}_{2s}$ arriving to:

Proposition 1.5.10. *The maps $\mathbb{S}_{wrx}$, $\mathbb{S}_{2wrx}$, $\mathbb{S}_{swrx}$ $\mathbb{S}_{2swrx}$ are all continuous.*

Again by remark 1.5.1 we obtain

Corollary 1.5.11. *The maps $\mathbb{S}_{rx}$, $\mathbb{S}_{2rx}$, $\mathbb{S}_{srx}$ $\mathbb{S}_{2srx}$ are all continuous.* [12]

$\diamond$ **The case $\mathbb{F}\subset H$:**

Corollary 1.5.12. *Let $\mathbb{F}\subset H$ be a finite-dimensional subspace and let $\alpha = \{e_i \mid i = 1,\,\ldots,\,p\}$ be a basis for $\mathbb{F}$. Let also $\mathcal{V} := \{v_b\in L^\infty([0,\,T],\,\mathbb{F}) \mid b\in B\}$ be a uniformly l_1-bounded family of controls, say $|v_b|_{L^\infty([0,\,T],\,(\mathbb{R}^p,\,l_1))}\le M$ for a constant $M>0$ independent of the parameter b. Then the map*

$$(u_0,\,F,\,v,\,\nu)\mapsto\mathbb{S}(u_0,\,F,\,v,\,\nu)$$

[12] These "rx"-maps are defined similarly as the "wrx" ones, just considering the "rx"-topology in the factor of essentially bounded functions.

is (X, Y)*-continuous. Here* $\mathbb{S}(u_0, F, v, \nu)$ *is the solution of the equation for the given data and the pair* (X, Y) *is one of the following*

$$(H \times L^2(0, T, V') \times \mathcal{V}_{rx} \times \mathbb{R}^+, Y_1); \quad (V \times L^2(0, T, H) \times \mathcal{V}_{rx} \times \mathbb{R}^+, Y_2);$$

where

$$Y_1 \in \{L^2(0, T, V), C([0, T], H)\}; \quad Y_2 \in \{L^2(0, T, D(A)), C([0, T], V)\}.$$

Proof. Let $\varepsilon > 0$ be a real number. Set $f_i \in D(A)$ such that $|e_i - f_i| < \varepsilon$; for any $v_b = \sum_{i=1}^p v_b^i e_i \in \mathcal{V}$ define $w_b = \sum_{i=1}^p v_b^i f_i$. Note that the family $\beta = \{f_i \mid i = 1, \ldots, p\}$ is linearly independent for small enough ε and that, $|v_b - w_b|_{L^\infty(0, T, H)}$ (and so also $|v_b - w_b|_{L^\infty(0, T, V')}$) is small if so is ε. Indeed for $t \in [0, T]$, $|v_b(t) - w_b(t)|$ is bounded by $\sum_{i=1}^p M|e_i - f_i| \leq M\varepsilon p$.

For a target space Y, suitable for the data, we have:

$$\begin{aligned}
&|\mathbb{S}(u_0, F, v_b, \nu) - \mathbb{S}(u_0, F, v_a, \nu)|_Y \\
\leq &|\mathbb{S}(u_0, F, v_b, \nu) - \mathbb{S}(u_0, F, w_b, \nu)|_Y + |\mathbb{S}(u_0, F, w_b, \nu) - \mathbb{S}(u_0, F, w_a, \nu)|_Y \\
&+ |\mathbb{S}(u_0, F, w_a, \nu) - \mathbb{S}(u_0, F, v_a, \nu)|_Y
\end{aligned}$$

and, since $|w_b - w_a|_{rx(\beta)} = |v_b - v_a|_{rx(\alpha)}$ we have that, for small ε and small $|v_b - v_a|_{rx(\alpha)}$, the norm

$$|\mathbb{S}(u_0, F, v_b, \nu) - \mathbb{S}(u_0, F, v_a, \nu)|_Y$$

is small. $\square$

Chapter 2

Saturating sets

2.1 V-saturating sets

We have been changing the definition of **saturating set** trying to make it more flexible. At the starting point in [4], for the cases of periodic boundary conditions and in [46], for the case of a Rectangle with so-called Lions boundary conditions, to the definition of saturating set g was associated a sequence $(H^j)_{n\in\mathbb{N}}$ of subspaces of $D(A)$ satisfying:

1. $H^0 := span(g)$;
2. $H^{j+1} \subseteq \Big(H^j + Conv\{-BY \mid Y \in H^j\}\Big) \cap \Big(H^j - Conv\{-BY \mid Y \in H^j\}\Big)$;
3. $\overline{\bigcup_{i\in\mathbb{N}} H^i} = H$ and;
4. there exists a finite subset $\mathcal{H}^j \subseteq \mathbb{N}_0$ such that $H^j = span\{W_k \mid k \in \mathcal{H}^j\}$ for all $j \in \mathbb{N}$; W_k are eigenfunctions of the Laplacean operator.

Finding such an increasing sequence of subspaces, spanned by eigenfunctions, was possible in [4, 46] due to the particularity of the cases treated there. Changing either the domain or the boundary conditions such sequence mail fail to increase (strictly). Since for the method we are going to introduce in the next chapter, we need a strictly increasing sequence, we concluded that the definition should be relaxed. The first step we could do, in that direction, was do not ask for the spanning of a finite number of eigenfunctions, i.e., in the recursive step we just take the maximal subspace of $D(A)$:

$$G^{j+1} := \Big(G^j + Conv\{BY \mid Y \in G^j\}\Big) \cap \Big(G^j - Conv\{BY \mid Y \in G^j\}\Big) \cap D(A)$$

but, even this generated sequence G^j turned out to be too tight. It was necessary to relax more; finally we have arrived to the following definition, that seems to be flexible enough to generate a strictly increasing sequence of subspaces.

Definition 2.1.1. *A finite set of vectors $g \subset V$ is said V-**saturating** if the sequence $(G^j)_{n\in\mathbb{N}}$ of finite dimensional subspaces of V defined recursively by*

1. $G^0 := span(g)$;
2. $G^{j+1} := \Big(G^j + Conv\overline{\{BY \mid Y \in G^j\} \cap V}\Big) \cap \Big(G^j - Conv\overline{\{BY \mid Y \in G^j\} \cap V}\Big)$

satisfies

$$\overline{\bigcup_{i\in\mathbb{N}} G^i} = H.$$

Here $\overline{\{BY \mid Y \in G^j\} \cap V}$ *stays for the closure of the intersection* $\{BY \mid Y \in G^j\} \cap V$ *in* H.

Remark 2.1.1. *The existence of a saturating set will be the sufficient conditions for the controllability results. Less flexible notions of saturating sets may give a sufficient condition for those results as well but, less flexible is the notion, more difficult (if possible) is to find the respective saturating set. That is why we have been trying to relax the best we can that notion.*

We have tried to relax even more the definition of saturating set (for example we tried to replace V *by* $L^4(T\Omega)\cap H$ *in the previous definition) but, with the generated sequences we have obtained, we could not apply our method to derive the controllability results.*

Remark 2.1.2. *Note that* $\{BY \mid Y \in G^j\} \cap V$ *and its closure are cones so, also*

$$Conv\overline{\{BY \mid Y \in G^j\} \cap V}$$

is a convex cone and G^{j+1} *is a linear space.*

Remark 2.1.3. *In the previous definition,* Π_j *being the orthogonal projection onto* G^j*,the conditions*

$$\begin{aligned} &(A) \quad \overline{\bigcup_{i\in\mathbb{N}} G^i} = H;\\ &(B) \quad \forall x \in H\, [j \to \infty \text{ only if } |x - \Pi_j x| \to 0]; \end{aligned}$$

are equivalent.

Remark 2.1.4. *The linear space* G^{j+1} *is contained in*

$$G_{j+1} := span\{\gamma_n,\, B(\gamma_n,\, \gamma_m) + B(\gamma_m,\, \gamma_n) \mid\; n,\, m = 1,\, \ldots,\, r\}$$

where $\{\gamma_n \mid n = 1,\, \ldots,\, r\}$ *is any basis for* G^j*: for is enough to check that* BG^j *is contained in* $span\{B(\gamma_n,\, \gamma_m) + B(\gamma_m,\, \gamma_n) \mid\; n,\, m = 1,\, \ldots,\, r\}$. *Write* $X \in G^j$ *as* $X = \sum_{i=1}^{r} X_i\gamma_i$ *then*

- $B(X_1\gamma_1) = X_1^2 B\gamma_1 = X_1^2 B(\gamma_1,\, \gamma_1) \in G_{j+1}$ *and;*
- *if* $B\Big(\sum_{i=1}^{p} X_i\gamma_i\Big) \in G_{j+1}$ *and* $1 \le p \le r-1$, *then*

$$\begin{aligned} B\left(\sum_{i=1}^{p+1} X_i\gamma_i\right) &= B\left(\sum_{i=1}^{p} X_i\gamma_i\right) + X_{p+1}^2 B\gamma_{p+1}\\ &\quad + \sum_{i=1}^{p} X_{p+1}X_i\Big(B(\gamma_{p+1},\, \gamma_i) + B(\gamma_i,\, \gamma_{p+1})\Big) \end{aligned}$$

  so, $B\Big(\sum_{i=1}^{p+1} X_i\gamma_i\Big) \in G_{j+1}$.

Therefore if G^j *is finite dimensional, then so is* G^{j+1} *and; in that case, if* $G^j \subseteq V$ *also* $G^{j+1} \subseteq V$.

2.2 l-saturating sets

Let $p \in \mathbb{N}_0$ be a positive natural number and let $g := \{U_i \mid i = 1, \ldots, p\} \subseteq V$ be a finite set satisfying $B(U_i) = 0$ for all $i \in \{1, \ldots, p\}$.

Put $L^0 := span(g)$. Then all the vectors $B(U_i \pm \lambda U_j) = \pm\lambda\big(B(U_i, U_j) + B(U_j, U_i)\big)$ are in $\{Bu \mid u \in L^0\}$. By remark 2.1.4 $span\{Bu \mid u \in L^0\}$ is contained in $span\{B(U_i, U_j) + B(U_j, U_i) \mid i, j = 1, \ldots, p\}$. Since

$$\begin{aligned} &span\{B(U_i, U_j) + B(U_j, U_i) \mid i, j = 1, \ldots, p\} \\ =&Conv\{\pm\lambda\big(B(U_i, U_j) + B(U_j, U_i)\big) \mid i, j = 1, \ldots, p,\ \lambda \in \mathbb{R}\} \\ \subseteq&Conv\{Bu \mid u \in L^0\}; \end{aligned}$$

we may conclude that $span\{Bu \mid u \in L^0\} = Conv\{Bu \mid u \in L^0\}$ and that

$$\begin{aligned} L^1 :=& \; span\{U_i,\, B(U_i, U_j) + B(U_j, U_i) \mid i, j = 1, \ldots, p\} \cap V \\ =& \Big(L^0 + (Conv\{BY \mid Y \in L^0\} \cap V)\Big) \cap \Big(L^0 - (Conv\{BY \mid Y \in L^0\} \cap V)\Big). \end{aligned}$$

Now given a finite linear subspace $L^m \subseteq V$ containing L^0, $v \in L^m$ such that $Bv \in V$, $U_i \in g$, $\lambda \in \mathbb{R}$ and $k \in \mathbb{N}_0$, the vector field

$$B(\lambda k U_i \pm \frac{1}{k} v) = \frac{1}{k^2} Bv \pm \lambda\big(B(U_i, v) + B(v, U_i)\big)$$

belongs to $B(L^m)$ and $\pm\lambda\big(B(U_i, v) + B(v, U_i)\big) \in \overline{B(L^m)}$. Clearly we have that $\frac{1}{k^2}Bv \pm \lambda\big(B(U_i, v) + B(v, U_i)\big) \in V$ for any k if, and only if, the limit $\pm\lambda\big(B(U_i, v) + B(v, U_i)\big)$ is in V.

In particular

$$\begin{aligned} L^{m+1} :=& \; L^m + \Big(span\{-B(U_i, v) - B(v, U_i) \mid i = 1, \ldots, p,\ v \in L^m,\ Bv \in V\} \cap V\Big) \\ \subseteq& \Big(L^m + Conv\overline{\{BY \mid Y \in L^m\} \cap V}\Big) \cap \Big(L^m - Conv\overline{\{BY \mid Y \in L^m\} \cap V}\Big). \end{aligned}$$

Definition 2.2.1. *A finite set $g = \{U_i \mid i = 1, \ldots, p\} \subset V$, satisfying $B(U_i) = 0$ for all $i \in \{1, \ldots, p\}$, is said* l**-saturating** *if the sequence $(L^j)_{n \in \mathbb{N}}$ of finite dimensional subspaces of H defined recursively by*

1. $L^0 := span(g)$;
2. $L^{m+1} := L^m + span\{-B(U_i, v) - B(v, U_i) \mid i = 1, \ldots, p,\ v \in L^m,\ Bv \in V\} \cap V$

satisfies

$$\overline{\bigcup_{i \in \mathbb{N}} L^i} = H.$$

We have seen that the V-saturating sequence (G^m) of definition 2.1.1 relative to the set $g = \{U_i \mid i = 1, \ldots, p\}$ of vector fields U_i, with $B(U_i) = 0$, satisfy $L^m \subseteq G^m$ for all order m. Therefore any l-saturating set is in particular V-saturating.

Note that the definition of l-saturating set, like that of V-saturating set, does depend on the space V.

Chapter 3

Controllability

3.1 Technical lemmas

Definition 3.1.1. *A sequence of probabilistic Radon measures μ^j in $\mathbb{R}^m$ is said to* **converge weakly** *to the measure μ if, for any continuous function g in $\mathbb{R}^m$ with compact support, we have*

$$\langle \mu^j,\, g\rangle \to \langle \mu,\, g\rangle \quad as \quad j \to +\infty.$$

By $\langle \mu^j,\, g\rangle$ we mean $\int_{\mathbb{R}^m} g(z)d\mu^j(z)$. [1]

Definition 3.1.2. *A* **generalized control** *in $U \subset \mathbb{R}^m$ is a weakly measurable family μ_t of Radon probabilistic measures concentrated on U.*

By weakly measurability we mean that $h(t) = \int_{\mathbb{R}^m} g(t,\, u)\, d\mu_t(u)$ is Lebesgue measurable, for all functions $g(t,\, u)$ continuous in $(t,\, u) \in \mathbb{R}^{1+m}$ and such that, for fixed t, $g(t,\, u)$ is compactly supported in $\mathbb{R}^m$.

An ordinary control $v(t)$ may be seen as the family of Radon measures $\bar{\delta}_{v(t)}$: for a fixed t we have the Dirac measure concentrated at $v(t)$.

Definition 3.1.3. *A sequence of generalized controls μ_t^j is said to* **converge weakly** *to a generalized control μ_t if, for any continuous function $g(t,\, u)$ in $\mathbb{R}^{1+m}$, with compact support in the variable $u \in \mathbb{R}^m$, we have*

$$\int_{\mathbb{R}} \langle \mu_t^j,\, g(t,\, u)\rangle\, dt \to \int_{\mathbb{R}} \langle \mu_t,\, g(t,\, u)\rangle\, dt \quad as \quad j \to +\infty.$$

A sequence of generalized controls μ_t^j is said to **converge strongly** *to a generalized control μ_t if*

$$\int_{\mathbb{R}} \|\mu_t^j - \mu_t\|_\sigma\, dt \to 0 \quad as \quad j \to +\infty.$$

Where the **strong norm** *$\|\mu\|_\sigma$ of the measure μ is defined by*

$$\|\mu\|_\sigma = \sup\{\langle \mu,\, g\rangle \mid |g|_{C^0} \le 1,\, g \text{ has compact support in } u\}.$$

[1] Recall that a Radon probabilistic measure μ is linear continuous functional, in the space of compactly supported continuous functions, satisfying $\mu(\zeta) = 1$ for $\zeta \equiv 1$.

Lemma 3.1.1 ([28], ch.2). *Let $\mu_t^j(b)$ be a sequence of generalized controls converging weakly to a generalized control $\mu_t(b)$ uniformly w.r.t. (with respect to) the parameter $b \in B$. Let all the measures $\mu_t^j(b)$ and $\mu_t(b)$ be concentrated in a single bounded set $N \subset \mathbb{R}^m$. Then for any continuous function $g(t, u))$ in $\mathbb{R} \times \mathbb{R}^m$ compactly supported in the variable u and all real numbers t_1, t_2 we have that*

$$\int_{[t_1, t_2]} \langle \mu_t^j(b), g(t, u) \rangle \, dt \to \int_{[t_1, t_2]} \langle \mu_t(b), g(t, u) \rangle \, dt \quad as \quad j \to +\infty$$

uniformly w.r.t. the parameter $b \in B$.

Lemma 3.1.2 (Approximation Lemma. [28], ch.3). *Let B be a metric space and $\{\mu_t(b), b \in B\}$ be a strongly continuous family of generalized controls. Let all the measures be concentrated in a single bounded set $N \subset \mathbb{R}^m$. Then there exist a sequence of piecewise constant ordinary controls $(\bar{\delta}_{u^i(t,b)})_{i \in \mathbb{N}_0}$ such that*

- *all the measures $\bar{\delta}_{u^i(t,b)}$ are concentrated on N, i.e., $u^i(t, b) \in N$;*
- *for fixed i, the family $\{\bar{\delta}_{u^i(t,b)} \mid b \in B\}$ is strongly continuous;*
- *$\bar{\delta}_{u^i(t,b)}$ converges weakly to $\mu_t(b)$ as $i \to +\infty$, uniformly w.r.t. the parameter b.*

From the proof of the previous lemma and from the "Remark on the Terminology" at the end of the chapter 3 in [28] we can derive the following corollary:

Corollary 3.1.3. *Let $N = \{q_1, q_2, \ldots, q_r\}$ be a finite set in $\mathbb{R}^m$. Let us be given a strongly continuous family $\{\lambda_t(b) = \sum_{j=1}^r \lambda^j(t, b)\bar{\delta}_{q_j} \mid b \in B\}$ such that $\sum_{j=1}^r \lambda^j(t, b) = 1$ and $\lambda^j(t, b) \geq 0$ for all $j \in \{1, 2, \ldots, r\}$. Then there exist a sequence of piecewise constant ordinary controls $\bar{\delta}_{u^i(t,b)}$ such that*

- *$u^i(t, b) \in N$;*
- *for fixed i, the family $\{\bar{\delta}_{u^i(t,b)} \mid b \in B\}$ is strongly continuous;*
- *$\bar{\delta}_{u^i(t,b)}$ converges weakly to $\mu_t(b)$ as $i \to +\infty$, uniformly w.r.t. the parameter b;*
- *for fixed i, the number of the intervals of constancy is the same for all $\bar{\delta}_{u^i(t,b)}$.*

Now consider a family of essentially bounded ordinary controls defined in $[0, T]$

$$\left\{ v(t, b) = \sum_{j=1}^r v^j(t, b) p_j \mid b \in B \right\}$$

taking values in the convexification $ConvN$, $N = -A \cup A$, $A = \{p_1, \ldots, p_r\}$.[2] The elements of A are supposed to be linearly independent. Necessarily we have that $\sum_{j=1}^r |v^j(t, b)| \leq 1$ because $v(t, b) \in Conv(-A \cup A)$.

Set $v_+^j(t, b) = \sup\{v^j(t, p), 0\}$, $v_-^j(t, b) = -\inf\{v^j(t, b), 0\}$ and

$$\eta(t, b) = 1 - \sum_{j=1}^r v_+^j(t, b) + v_-^j(t, b) = 1 - \sum_{j=1}^r |v^j(t, b)|.$$

[2] The controls defined in $[0, T]$ may be seen as controls defined in all the real line extending the former by 0 outside $[0, T]$.

Write $v(t,\, b)$ as

$$v(t,\, b) = \sum_{j=1}^{r} \left(v_+^j(t,\, b) + \frac{\eta(t,\, b)}{2r} \right) p_j + \left(v_-^j(t,\, b) + \frac{\eta(t,\, b)}{2r} \right) (-p_j)$$

and consider the family of generalized controls

$$\hat{v}(t,\, b) = \sum_{j=1}^{r} \left(v_+^j(t,\, b) + \frac{\eta(t,\, b)}{2r} \right) \bar{\delta}_{p_j} + \left(v_-^j(t,\, b) + \frac{\eta(t,\, b)}{2r} \right) \bar{\delta}_{-p_j}.$$

Let $\mathcal{A} = spanA$ be endowed with the norm $\sum_{j=1}^{r} |x^i|$, $x = \sum_{j=1}^{r} x^i p_i \in \mathcal{A}$.
Given $a,\, b \in B$ we have

$$\begin{aligned}
&\int_0^T \|\hat{v}(t,\, b) - \hat{v}(t,\, a)\|_\sigma \, dt \\
&= \int_0^T \sup \langle \hat{v}(t,\, b) - \hat{v}(t,\, a),\, g(t,\, u) \rangle \, dt \\
&= \int_0^T \sup \Bigg[\sum_{j=1}^{r} \left(\left(v_+^j(t,\, b) + \frac{\eta(t,\, b)}{2r} \right) - \left(v_+^j(t,\, a) + \frac{\eta(t,\, a)}{2r} \right) \right) g(t,\, p_j) \\
&\qquad + \left(\left(v_-^j(t,\, b) + \frac{\eta(t,\, b)}{2r} \right) - \left(v_-^j(t,\, a) + \frac{\eta(t,\, a)}{2r} \right) \right) g(t,\, -p_j) \Bigg] \, dt;
\end{aligned}$$

where the supremum is to be taken over all continuous functions $g(t,\, u)$ compactly supported in u and with $|g(t,\, u)|_{C^0} = 1$.

Setting an essentially bounded function $\tilde{g}(t,\, u)$ such that $\tilde{g}(t,\, \pm p_j) = sign(v_\pm^j(t,\, b) - v_\pm^j(t,\, a))$, we may conclude that

$$\begin{aligned}
&\int_0^T \|\hat{v}(t,\, b) - \hat{v}(t,\, a)\|_\sigma \, dt \\
&= \int_0^T \sup \Bigg[\sum_{j=1}^{r} \left| \left(v_+^j(t,\, b) + \frac{\eta(t,\, b)}{2r} \right) - \left(v_+^j(t,\, a) + \frac{\eta(t,\, a)}{2r} \right) \right| \\
&\qquad + \left| \left(v_-^j(t,\, b) + \frac{\eta(t,\, b)}{2r} \right) - \left(v_-^j(t,\, a) + \frac{\eta(t,\, a)}{2r} \right) \right| \Bigg] \, dt. \qquad (3.1)
\end{aligned}$$

Note that we may define an essentially bounded function $\tilde{g}(t,\, u)$ taking the value $\tilde{g}(t,\, \pm p_j)$ for all u in small neighborhoods of each $\pm p_j$ and vanishing outside these neighborhoods. Such a function can be approximated in $L^1(]0,\, T[\times\mathbb{R}^r)$ by continuous functions f^n, compactly supported in u, with values in $[0, 1]$. For each $\pm p_j$ the sequence $f^n(t,\, \pm p_j)$ will converge to $\tilde{g}(t,\, \pm p_j)$ in $L^1(]0,\, T[)$.

If the family $\{v(t,\, b) \mid,\, b \in B\}$ is parameterized continuously in $L^1(0,\, T,\, \mathcal{A})$-norm, we have that the coordinates $v^j(t,\, b)$ go to $v^j(t,\, a)$ in $L^1(0,\, T,\, \mathbb{R})$ as b go to a in B. Thus also $v_\pm^j(t,\, b)$ go to $v_\pm^j(t,\, a)$ and; $\eta(t,\, b)$ go to $\eta(t,\, a)$ in $L^1(0,\, T,\, \mathbb{R})$ as b go to a in B.

Therefore from (3.1) we may conclude that

$$\int_0^T \|\hat{v}(t,\, b) - \hat{v}(t,\, a)\|_\sigma \, dt \to 0 \quad \text{iff} \quad |v(t,\, b) - v(t,\, a)|_{L^1(0,\, T,\, \mathcal{A})} \to 0. \tag{3.2}$$

By corollary 3.1.3, the family $\hat{v}(t,\, b)$ may be weakly approximated, uniformly w.r.t. the parameter b, by a family of piecewise constant ordinary controls

$$\bar{\delta}_{v_i(t,\, b)} \equiv \sum_{j=1}^{r} v^j_{+i}(t,\, b) p_j + v^j_{-i}(t,\, b)(-p_j); \quad v^j_{\pm i}(t,\, b) \in \{0,\, 1\},\, \sum_{j=1}^{r} v^j_{\pm i}(t,\, b) = 1,$$

taking values in N and, by lemma 3.1.1, the integral $\int_{[t_1,\, t_2]} \langle \hat{v}(t,\, b) - \bar{\delta}_{v_i(t,\, b)},\, f(t,\, u)\rangle \, dt$ goes to 0 as $i \to +\infty$, uniformly w.r.t. the parameter $b \in B$, for all continuous functions f with $|f|_{C^0} \leq 1$, compactly supported in u.

In particular setting a compactly supported continuous function $\tilde{f}(t,\, u)$ coinciding with $\frac{\pm p_j}{|p_j|_{\mathbb{R}^m}}$ for $(t,\, u) = (t,\, \pm p_j)$ and; writing $v^j_i := v^j_{+i} - v^j_{-i}$ and $v^j := v^j_+ - v^j_-$, we have that

$$\begin{aligned}
\int_{[t_1,\, t_2]} \langle \hat{v}(t,\, b) - \bar{\delta}_{v_i(t,\, b)},\, \tilde{f}(t,\, u)\rangle \, dt &= \int_{[t_1,\, t_2]} \sum_{j=1}^{r} (v^j(t,\, b) - v^j_i(t,\, b)) \frac{p_j}{|p_j|_{\mathbb{R}^m}} \, dt \\
&= \sum_{j=1}^{r} \left(\int_{[t_1,\, t_2]} (v^j(t,\, b) - v^j_i(t,\, b)) \, dt \right) \frac{p_j}{|p_j|_{\mathbb{R}^m}}
\end{aligned}$$

must go to 0 as $i \to +\infty$, uniformly w.r.t. the parameter $b \in B$. Therefore also the "coordinates" $\int_{[t_1,\, t_2]} (v^j(t,\, b) - v^j_i(t,\, b)) \, dt$ must go to 0, i.e., the family $v_i(t,\, b)$ converges uniformly to the family $v(t,\, b)$ in β-relaxation metric with the basis $\beta = (p_1,\, \dots,\, p_r)$.

Definition 3.1.4. *We define δ-**metric** on the product space $\big(L^\infty([0,\, T], \mathbb{R}^d)\big)^2$ as*

$$\delta\big(u,\, v\big) := measure\{t \in [0,\, T] \mid u(t) \neq v(t)\}.$$

Remark 3.1.1. *The double 2δ of the δ-**metric** is the restriction, to the space of essentially bounded ordinary controls, of the strong convergence metric defined on the space of relaxed (generalized) controls. Indeed for ordinary controls $u(t)$, $v(t)$ taking values in $\mathbb{R}^m$,*

$$\int_0^T \sup \langle \bar{\delta}_{u(t)} - \bar{\delta}_{v(t)},\, g(t,\, u)\rangle \, dt = \int_0^T \sup[g(t,\, u(t)) - g(t,\, v(t))] \, dt$$

and, setting a compactly supported piecewise continuous and bounded function $g(t,\, u)$, taking the value 1 in the bounded graph $\{(t,\, u) \mid u = u(t)\}$ and the value -1 in $\{(t,\, u) \mid u = v(t) \neq u(t)\}$ – such a function may by approximated by continuous functions in $L^1(\mathbb{R}^{1+m})$-norm – we may conclude that $\|\bar{\delta}_{u(t)} - \bar{\delta}_{v(t)}\|_\sigma = 2\delta(u,\, v)$.

Remark 3.1.2. *The δ-metric and L^q-metric, $0 < q < +\infty$, give equivalent topologies in the subset of piecewise constant functions taking values on a fixed finite set $S = \{p_1,\, \dots,\, p_s\}$, because*

$$m(\delta(f,\, g))^{1/q} \leq |f - g|_{L^q(0,\, T,\, \mathcal{A})} \leq M(\delta(f,\, g))^{1/q};$$

for $m = \min |p_i - p_j|$, $M = \max |p_i - p_j|$ with $p_i \neq p_j$ and where p_i and p_j vary in S.

Moreover in that subset, when the number of intervals of constancy is the same, the δ-continuity of a family of controls is equivalent to the "continuity of the lengths" of the intervals of constancy of the controls in the case all the functions of the family assume the (same) value $p_{i(k)}$ in the k^{th} interval of constancy (for a given function of the family some of the intervals of constancy may degenerate to a single point).

Therefore we may conclude the following:

Lemma 3.1.4. *Let $A := \{p_1, p_2, \ldots, p_r\}$ be linearly independent in $\mathbb{R}^d$ and*

$$\mathcal{V} := \{v(t, b) \in L^\infty([0, T], Conv(-A \cup A)) \mid b \in B\}$$

be a L^1-continuous family of $Conv(-A \cup A)$-valued functions. Then for each $\varepsilon > 0$ one can construct a family $\mathcal{V}^\varepsilon := \{v^\varepsilon(t, b) \in L^\infty([0, T], -A \cup A) \mid b \in B\}$ of $(-A \cup A)$-valued functions such that

- *$\mathcal{V}^\varepsilon$ is L^q-continuous, i.e., $b \mapsto v^\varepsilon(\cdot, b)$ is $(B, L^q(0, T, -A \cup A))$-continuous;*
- *$\mathcal{V}^\varepsilon$ ε-approximates, uniformly w.r.t. b, the family $\mathcal{V}$ in relaxation metric, i.e., $\forall b \in B, \quad |v^\varepsilon(\cdot, b) - v(\cdot, b)|_{rx(\beta)} < \varepsilon$ with $\beta = (p_1, p_2, \ldots, p_r)$ and;*
- *the elements of $\mathcal{V}^\varepsilon$ are piecewise constant and the number of intervals of constancy is the same for all $b \in B$.*

It is said that the intervals of constancy can be taken the same for all $b \in B$ but, proceeding as in [28, 27]; some of those intervals may degenerate to a single point.[3] We claim that we can suppose non-degeneracy of the intervals.[4] We may even suppose that there exists a lower bound θ^ε for the lengths of the intervals of constancy of the family $\mathcal{V}^\varepsilon$, i.e., for all $b \in B$ none of the $v^\varepsilon(\cdot, b)$ has an interval of constancy with length less than θ^ε. We have a modified version of this lemma (the only difference is the addiction of the last item in the corollary):

Corollary 3.1.5 (modified Approximation lemma)**.** *With A and $\mathcal{V}$ as in the lemma 3.1.4, for each $\varepsilon > 0$ there exist a real number $\theta^\varepsilon > 0$ and a family*

$$\mathcal{Z}^\varepsilon := \{z^\varepsilon(t, b) \in L^\infty([0, T], -A \cup A) \mid b \in B\}$$

of $(-A \cup A)$-valued functions such that

- *$\mathcal{Z}^\varepsilon$ is L^q-continuous, $0 < q < +\infty$;*
- *$\forall b \in B \quad |z^\varepsilon(\cdot, b) - v(\cdot, b)|_{rx(\beta)} < \varepsilon$;*
- *the elements of $\mathcal{Z}^\varepsilon$ are piecewise constant and the number of intervals of constancy is the same for all $b \in B$ and;*
- *for all $b \in B$ all the intervals of constancy of $z^\varepsilon(\cdot, b)$ have a length not less than $\theta^\varepsilon > 0$.*

[3] In a suitable interval T_i the length of the interval of constancy where we apply constant control p_j is given by expressions $\int_{I_i} \left(v_+^j(t, b) + \frac{\eta(t, b)}{2r} \right) dt$; I_i is a subinterval of $[0, T]$ depending of the order i of the approximation $\bar{\delta}_{v_i}$.

[4] Note that it is not enough to eliminate the degenerate intervals because the number of intervals would not be the same for all $b \in B$.

The corollary follows from the previous lemma 3.1.4 and from the following: (see [48, section 4.10.2] for details):

Lemma 3.1.6. *Given $K > 0$, $\gamma > 0$ and $L \in \mathbb{N}_0$. Define the sets*

$$\mathcal{P}_0 := \{(x_1, \ldots, x_L) \in \mathbb{R}^L \mid x_i \geq 0, \sum_{i=1}^{L} x_i = K\}$$
$$\mathcal{P}_\theta := \{(x_1, \ldots, x_L) \in \mathbb{R}^L \mid x_i \geq \theta, \sum_{i=1}^{L} x_i = K\}$$

where $\theta > 0$. Choose $n \in \mathbb{N}_0$ such that $\frac{(L+1)K}{nL} < \gamma$ and put

$$\theta = \frac{K}{nL}. \quad [5] \tag{3.3}$$

Then the map $P_{0,\theta} = (P^1_{0,\theta}, \ldots, P^L_{0,\theta})$, defined on $\mathcal{P}_0$ by:

$$P^i_{0,\theta}(x) := \big(1 - \frac{1}{n}\big)\big(x_i - \frac{K}{L}\big) + \frac{K}{L},$$

is continuous, take its values on $\mathcal{P}_\theta$ and, satisfies $|P^i_{0,\theta}(x) - x_i| < \gamma$.

To each piecewise constant control of the family $\{z(\cdot, b) \mid b \in B\}$ will be then, associated a partition $X(b)$ of $[0, T]$ into L non-degenerated intervals of lengths $x_i \geq \theta > 0$:

$$X(b) = (x_1, x_2, \ldots, x_L) \in \mathbb{R}^L,\ L \in \mathbb{N}_0,\ \sum_{i=1}^{L} x_i = T,$$

where L and θ are independent of the parameter b. We put

$$A(b) = \{(0 = \alpha_0, \alpha_1, \ldots, \alpha_L = T)\}$$

for the end points of the intervals in $X(b)$. So,

$$A(b) \in \mathcal{A}_\theta := \{(\alpha_0, \alpha_1, \ldots, \alpha_L) \in \mathbb{R}^{L+1} \mid \alpha_0 = 0, \alpha_L = T,$$
$$\alpha_i - \alpha_{i-1} \geq \theta, \sum_{i=1}^{L}(\alpha_i - \alpha_{i-1}) = T\}.$$

Another lemma we will need is

Lemma 3.1.7. *For any $w \in \mathbb{R}$, $\quad w \geq 3$ and $A = (\alpha_0, \alpha_1, \ldots, \alpha_L) \in \mathcal{A}_\theta$ we can construct a function $\phi^w(\cdot, A) \in W^{1,\infty}([0, T], \mathbb{R})$ with the following properties:*

- *$\phi^w(\cdot, A)$ vanishes at the points α_i, $\quad i = 0, \ldots, L$;*
- *$\phi^w(\cdot, A) \in W^{1,\infty}([0, T], \mathbb{R})$ with*

$$|\phi^w(\cdot, A)|_{C([0,T],\mathbb{R})} \leq 1; \qquad |\dot{\phi}^w(\cdot, A)|_{L^\infty([0,T],\mathbb{R})} \leq \frac{w(1+\theta)}{\theta}$$

[5] So, θ depends on both K, L and γ.

- $\delta(\phi^w(t,\, A),\, \sin(wt)) \le \frac{2T}{w}$ *and;*
- *For fixed* w, *the map* $\Phi_w : A \mapsto \phi^w(\cdot,\, A)$ *is* $(\mathcal{A}_\theta,\, W^{1,2}(0,\, T,\, \mathbb{R}))$*-continuous (where* $\mathcal{A}_\theta$ *is endowed with the topology induced by the usual one of* $\mathbb{R}^{L+1}$*).*

Proof. For each $i \in \{1,\, \ldots,\, L\}$ put $x_i := \alpha_i - \alpha_{i-1}$ and $\rho_i = \frac{x_i}{w}$. Then subdivide each interval $[\alpha_{i-1},\, \alpha_i]$ as

$$[\alpha_{i-1},\, \alpha_i] = [\alpha_{i-1},\, \alpha_{i-1} + \rho_i[\cup [\alpha_{i-1} + \rho_i,\, \alpha_i - \rho_i] \cup]\alpha_i - \rho_i,\, \alpha_i].$$ [6]

In each interval $[\alpha_{i-1},\, \alpha_i]$, $i = 1,\, \ldots,\, L$, put

$$\phi^w(t,\, A) = \begin{cases} \frac{\sin(w(\alpha_{i-1}+\rho_i))}{\rho_i}(t - \alpha_{i-1}) & \text{if} \quad t \in [\alpha_{i-1},\, \alpha_{i-1} + \rho_i]; \\ \sin(wt) & \text{if} \quad t \in [\alpha_{i-1} + \rho_i,\, \alpha_i - \rho_i]; \\ \frac{\sin(w(\alpha_i - \rho_i))}{-\rho_i}(t - \alpha_i) & \text{if} \quad t \in [\alpha_i - \rho_i,\, \alpha_i]. \end{cases}$$

Then the graph of the restriction of $\phi^w(\cdot,\, A)$ to an interval $[\alpha_{i-1},\, \alpha_i]$ is a concatenation of a straight line, a piece of the graph of $\sin(wt)$ and another straight line. From the construction is clear that $\phi^w(\cdot,\, A)$ vanishes at the points α_i, $\quad i = 0,\, \ldots,\, L$ and that $\phi^w(\cdot,\, A)$ is continuous with $\|\phi^w(\cdot,\, A)\|_{C([0,T],\mathbb{R})} \le 1$.

In the subintervals $]\alpha_{i-1} + \rho_i,\, \alpha_i - \rho_i[$ we have $\dot{\phi}^w(t,\, A) = w\cos(wt)$ so, $|\dot{\phi}^w(t,\, A)| \le w \le \frac{w(1+\theta)}{\theta}$. In the subintervals $]\alpha_{i-1},\, \alpha_{i-1} + \rho_i[$ and $]\alpha_i - \rho_i,\, \alpha_i[$ we have $|\dot{\phi}_w(t,\, A)| \le \frac{1}{\rho_i} = \frac{w}{x_i} \le \frac{w}{\theta} \le \frac{w(1+\theta)}{\theta}$. Hence we have $\|\dot{\phi}_w(\cdot,\, A)\|_{L^\infty([0,T],\mathbb{R})} \le \frac{w(1+\theta)}{\theta}$. Therefore $\phi^w(\cdot,\, A) \in W^{1,\infty}([0,\, T],\, \mathbb{R})$ and

$$|\phi^w(\cdot,\, A)|_{W^{1,\infty}([0,T],\mathbb{R})} \le 1 + \frac{w(1+\theta)}{\theta}.$$

We see that $\phi^w(t,\, A)$ differs from $\sin(wt)$ only in the intervals $[\alpha_{i-1},\, \alpha_{i-1} + \rho_i[$ and $]\alpha_i - \rho_i,\, \alpha_i]$ so,

$$\delta(\phi^w(t,\, A),\, \sin(wt)) = \sum_{i=1}^{L} 2\rho_i = \sum_{i=1}^{L} 2\frac{x_i}{w} \le \frac{2T}{w}.$$

It remains to check the continuity property. That is not difficult and follows by direct computation but, since it is a bit long, we will not present it here. Anyway the computation can be found in the preprint [48]. $\square$

Now from the $(B,\, \mathcal{A}_\theta)$-continuity of the map $b \mapsto A(b)$ (which is equivalent to the δ-continuity of the family $\mathcal{Z}$) and, from the $(\mathcal{A}_\theta,\, W^{1,2})$-continuity of Φ_w we have the following:

Corollary 3.1.8. *For fixed* $w \ge 3$, *the map* $b \mapsto \phi^w(\cdot,\, b) := \phi^w(\cdot,\, A(b))$ *is* $(B,\, W^{1,2}(0,\, T,\, \mathbb{R}))$*-continuous.*

3.2 Comparing drivings

Let $g \subset V$ be a finite set of vector fields and, let $(G^j)_{j\in\mathbb{N}}$ be the sequence of subspaces of V defined, as in the definition of V-saturating set, recursively by:

[6] Note that, since $w \ge 3$ we have $\rho_i \le \frac{x_i}{3}$ and the subdivision is well defined.

1. $G^0 := span(g)$;
2. $G^{j+1} := \Big(G^j + Conv\overline{\{BY \mid Y \in G^j\} \cap V}\Big) \cap \Big(G^j - Conv\overline{\{BY \mid Y \in G^j\} \cap V}\Big).$

3.2.1 The family taking values on G^k

Let B be a subset of a normed space and $\Gamma := \{\gamma(t, b) \in L^\infty(0, T, G^k) | \, b \in B\}$, with $k \geq 1$, be a family of controls such that:

- Γ is equibounded w.r.t. b and t, say $|\gamma(t, b)| \leq M$ for some constant M and for all $(t, b) \in [0, T] \times B$;
- Γ is L^2-continuously parameterized in b: the map $b \mapsto \gamma(t, b)$ is $(B, L^2(0, T, H))$-continuous.

Using the fact that $-\overline{\{BY \mid Y \in G^{k-1}\} \cap V}$ is a cone we have that

$$\begin{aligned} G^k &= \Big(G^{k-1} + Conv\overline{\{BY \mid Y \in G^{k-1}\} \cap V}\Big) \cup \Big(G^{k-1} - Conv\overline{\{BY \mid Y \in G^{k-1}\} \cap V}\Big) \\ &\subseteq G^{k-1} - Conv\overline{\{BY \mid Y \in G^{k-1}\} \cap V} \\ &= Conv\Big(G^{k-1} - \overline{\{BY \mid Y \in G^{k-1}\} \cap V}\Big) \end{aligned}$$

so, for a give basis $\{E_i \mid i = 1, \ldots, s_0\}$ of G^k we have that

$$E_i = \sum_{p=1}^{P_i} \lambda_{i,p}(e_{i,p} - f_{i,p});$$

where $e_{i,p} \in G^{k-1}$; $f_{i,p} \in \overline{\{BY \mid Y \in G^{k-1}\} \cap V}$; $\lambda_{i,p} \geq 0$ and; $\sum_{p=1}^{P_i} \lambda_{i,p} = 1$.

Then $Conv\{\pm E_i \mid i = 1, \ldots, s_0\} \subseteq Conv(-R \cup R)$ with

$$R := \{e_{i,p} - f_{i,p} \mid \; i = 1, \ldots, s_0; \, p = 1, \ldots, P_i\}$$

and necessarily we may select from R a linearly independent set

$$A := \Big\{\hat{e}_r - \hat{f}_r, \, r = 1, \ldots, s_0\Big\}$$

for G^k. Note that some of the $\hat{e}_r \in G^{k-1}$ or $\hat{f}_r \in \overline{\{BY \mid Y \in G^{k-1}\} \cap V}$ may vanish.

Since the family of controls is uniformly bounded we have that for some constant $\Xi > 0$

$$\gamma(t, b) \in \Xi Conv(-A \cup A)$$

and, since the elements $-\hat{e}_r + \hat{f}_r$ of $-A \in G^k$, belong to $G^{k-1} - \overline{\{BY \mid Y \in G^{k-1}\} \cap V}$, we have that those elements may be written as $-\hat{e}_r + \hat{f}_r = \hat{e}_{s_0+r} - \hat{f}_{s_0+r}$, $r = 1, \ldots, s_0$.

Therefore, for some constant $\Xi > 0$

$$\gamma(t, b) \in Conv(-\Xi A \cup \Xi A); \quad -A \cup A = \Big\{\hat{e}_r - \hat{f}_r, \, r = 1, \ldots, s = 2s_0\Big\}.$$

3.2.2 Lowering the dimension

Relaxation. By corollary 3.1.5 we may approximate $\gamma(\cdot,\, b)$ by a piecewise constant control $\tilde\gamma(\cdot,\, b)$ such that $\tilde\gamma(\cdot,\, b)$ takes values in $\Xi\{\hat e_r - \hat f_r \mid r = 1,\, \dots,\, s\}$ and the family $\tilde\gamma(\cdot,\, b)$ approximates $\gamma(\cdot,\, b)$ in relaxation metric.

Fix $u_0 \in V$. The continuity of the equation in relaxation metric implies that, for $\tilde\gamma(\cdot,\, b)$ close to $\gamma(\cdot,\, b)$ in relaxation metric, the (strong) solutions u and $\tilde u$ of the systems

$$u_t = -\nu A u - Bu + \nu C u + F + \gamma(t,\, b), \quad u(0) = u_0$$

and

$$\tilde u_t = -\nu A \tilde u - B\tilde u + \nu C \tilde u + F + \tilde\gamma(t,\, b), \quad \tilde u(0) = u_0$$

are close in $C([0,\, T],\, V)$.

Leaving the boundary. Let $[t_{j-1},\, t_j]$ be the j^{th} interval of constancy associated with some parameter b: if m is the number of intervals of constancy we have $t_j = t_j(b)$ for $0 \le j \le m$; $t_0(b) = 0$ and $t_m(b) = T$. Let $e_j - f_j$ be the value taken by $\tilde\gamma(t,\, b)$ in this interval, i.e., for some $1 \le r \le s$

$$\tilde\gamma(t,\, b) = e_j - f_j = \Xi(\hat e_r - \hat f_r), \quad t \in [t_{j-1},\, t_j].$$

Set $\tilde a_j$ in $G^{k-1} \subseteq V$ such that $B\tilde a_j \in V$ and $|B\tilde a_j - f_j|$ is small. The solution $\tilde v$ of the system

$$\tilde v_t = -\nu A\tilde v - B\tilde v + \nu C\tilde v + F + e_j - B\tilde a_j, \quad \tilde v(t_{j-1}) = \tilde v_{j-1}$$

is close to $\tilde u$ in $C([t_{j-1},\, t_j],\, V)$, if $\tilde v(t_{j-1})$ is close to $\tilde u(t_{j-1})$ in V.

Going to $D(A)$. Let $a_j \in D(A)$ such that $|a_j - \tilde a_j|_V$ is small. Then $|Ba_j - B\tilde a_j|_{V'}$ is small and the solution v of the system

$$v_t = -\nu A v - Bv + \nu C v + F + e_j - Ba_j, \quad v(t_{j-1}) = v_{j-1}$$

is close to $\tilde v$ in $C([t_{j-1},\, t_j],\, H)$, if $v(t_{j-1}) \in V$ is close to $\tilde v(t_{j-1}) \in V$ in H. Note that both v and $\tilde v$ are in $C([t_{j-1},\, t_j],\, V)$ but, since the constant controls are Ba and $B\tilde a$ are not necessarily close in H, we can not guarantee the closeness of v and $\tilde v$ on $C([t_{j-1},\, t_j],\, V)$.

Imitation. Consider the solution z of system

$$z_t = -\nu A z - Bz + \nu C z + F + e_j + \sqrt{2}\phi^w_t a_j, \quad z(t_{j-1}) = z_{j-1},$$

where ϕ^w_t is a sinus-like function vanishing at the end points of $[t_{j-1},\, t_j]$ (constructed in lemma 3.1.7), and change variables putting $y = z - \sqrt{2}\phi^w a_j$. Clearly z and y coincide at the end points t_{j-1} and t_j of the interval of constancy. Moreover y satisfies

$$y_t = -\nu A(y + \sqrt{2}\phi^w a_j) - B(y + \sqrt{2}\phi^w a_j) + \nu C(y + \sqrt{2}\phi^w a_j) + F + e_j, \quad y(t_{j-1}) = z_{j-1}.$$

It turns out that: if $y(t_{j-1})$ and $v(t_{j-1})$ belong to V and are close in H-norm, then for big enough w

$$y \quad \text{is close to} \quad v \qquad \text{in} \quad C([t_{j-1},\, t_j],\, H). \tag{3.4}$$

The control in G^{k-1}. Let $\tilde z$ be the solution of

$$\tilde z_t = -\nu A\tilde z - B\tilde z + \nu C\tilde z + F + e_j + \sqrt{2}\phi^w_t \tilde a_j, \quad \tilde z(t_{j-1}) = \tilde z_{j-1}.$$

We claim that, if $\tilde z(t_{j-1}) \in V$ is close to $z(t_{j-1}) \in V$ in H, then

$$\tilde{z} \text{ is close to } z \text{ in } C([t_{j-1},\, t_j],\, H). \tag{3.5}$$

Therefore, the end points $u(t_j)$ and $\tilde{z}(t_j)$ are both in V and close in H-norm if, the initial points $u(t_{j-1})$ and $\tilde{z}(t_{j-1})$, both in V, are close in H-norm.

3.2.3 The family taking values on G^{k-1}

Consider the family of controls

$$\tilde{Z} := \{\tilde{z}(\cdot,\, b) \in L^\infty(0,\, T,\, G^{k-1}) \mid b \in B\}$$

where the control $\tilde{z}(\cdot,\, b)$ takes the value $e_j + \sqrt{2}\phi_t^w \tilde{a}_j$ in the interval $[t_{j-1}(b),\, t_j(b)]$:

- $\tilde{Z}$ is clearly equibounded w.r.t. t and b and;
- the map $b \mapsto \tilde{z}(\cdot,\, b)$ is $(B,\, L^2(0,\, T,\, H))$-continuous.

Therefore as soon as we prove (3.4) and (3.5) we have:

Theorem 3.2.1. *Let B be a subset of a normed space and*

$$\Gamma := \{\gamma(t,\, b) \in L^\infty(0,\, T,\, G^k) | \, b \in B\}$$

be a family of controls such that:

- *Γ is equibounded w.r.t. b and t, say $|\gamma(t,\, b)| \leq M$ for some constant M and for all $(t,\, b) \in [0,\, T] \times B$;*
- *the map $b \mapsto \gamma(t,\, b)$ is $(B,\, L^2(0,\, T,\, H))$-continuous.*

Then for all $\varepsilon > 0$, there exists a family of controls

$$\tilde{Z} := \{\tilde{z}(\cdot,\, b) \in L^\infty(0,\, T,\, G^{k-1}) \mid b \in B\}$$

such that

- *$\tilde{Z}$ is equibounded w.r.t. t and b and;*
- *the map $b \mapsto \tilde{z}(\cdot,\, b)$ is $(B,\, L^2(0,\, T,\, H))$-continuous;*

and the solutions u and $\tilde{z}$ of the equations

$$\begin{aligned} u_t &= -\nu A u - B u + \nu C u + F + \gamma(t,\, b); \\ \tilde{z}_t &= -\nu A \tilde{z} - B \tilde{z} + \nu C \tilde{z} + F + \tilde{z}(\cdot,\, b); \\ \tilde{z}(0) &= u(0) = u_0 \in V; \end{aligned}$$

satisfy $|\tilde{z}(T) - u(T)| < \varepsilon$.

3.2.4 Proof of statement (3.5)

The statement follows from the following:

Proposition 3.2.2. *Given a continuous function ϕ taking values in V such that $\dot{\phi} = \phi_t \in L^2(t_i, t_f, H)$. Then there exists a solution of the system*

$$y_t = -\nu A(y+\phi) - B(y+\phi) + \nu C(y+\phi) + F; \qquad y(t_i) = y_{t_i} \in V.$$

This solution varies continuously in $C([t_i, t_f], H)$ and $L^2(t_i, t_f, V)$ when ϕ varies continuously in $C([t_i, t_f], V)$ and when the initial condition varies in H.

By the proposition, setting $t_i = t_{j-1}$ and $t_f = t_j$, the solutions z and $\tilde{z}$, appearing in (3.5) will be close in $C([t_{j-1}, t_j], H)$ (and in $L^2(t_{j-1}, t_j, V)$) for small $\|\tilde{a} - a\|$, small $|z(t_{j-1}) - \tilde{z}(t_{j-1})|$, and any $w > 0$. Take our sinus-like function $\phi^w(t)$ for ϕ in the proposition.

By construction $\phi^w(t)$ vanishes at the end points t_{j-1} and t_j. By the proposition $y = z - \sqrt{2}\phi^w a$ and $\tilde{y} = \tilde{z} - \sqrt{2}\phi^w \tilde{a}$ are close in $C([t_{j-1}, t_j], H)$ and in $L^2(t_{j-1}, t_j, V)$, if $y(t_i) = z(t_i) \in V$ and $\tilde{y}(t_i) = \tilde{z}(t_i) \in V$ are close in H. The statement follows from the identity

$$z - \tilde{z} = y - \tilde{y} + \sqrt{2}\phi^w(a - \tilde{a})$$

and from $\|\sqrt{2}\phi^w(a - \tilde{a})\| \leq \sqrt{2}\|a - \tilde{a}\|$.

Proof of proposition 3.2.2. The existence follows from the existence of a strong solution u in the intersection $C([t_i, t_f], V) \cap L^2(t_i, t_f, D(A))$ for the equation

$$u_t = -\nu Au - Bu + \nu Cu + F + \phi_t; \qquad u(t_i) = y(t_i).$$

$y = u - \phi$ is the wanted solution. Moreover, since $\phi \in C([t_i, t_f], V)$, we have $y \in C([t_i, t_f], V)$.

Multiplying

$$y_t = -\nu A(y+\phi) - B(y+\phi) + \nu C(y+\phi) + F$$

by y in H we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}|y|^2 \leq& -\nu\|y+\phi\|^2 + \nu\|y+\phi\|\|\phi\| + K|y+\phi|\|y+\phi\|\|\phi\| + K\|y+\phi\||y| + |F||y| \\
\leq& -\frac{\nu}{2}\|y+\phi\|^2 + D\Big(\|\phi\|^2 + |y+\phi|^2\|\phi\|^2 + |y|^2 + |F|^2\Big) \\
\leq& -\frac{\nu}{2}\|y+\phi\|^2 + D_1\Big(\|\phi\|^2 + |y|^2\|\phi\|^2 + \|\phi\|^4 + |y|^2 + |F|^2\Big).
\end{aligned}$$

Hence

$$|y|^2_{C([t_i, t_f], H)} \leq D_2\Big(|y(t_i)|^2 + |\phi|^2_{L^2(t_i, t_f, V)} + |\phi|^4_{L^4(t_i, t_f, V)} + |F|^2\Big)$$

and

$$|y|^2_{L^2(t_i, t_f, V)} \leq D_3\Big(|y(t_i)|^2 + |\phi|^2_{L^2(t_i, t_f, V)} + |\phi|^4_{L^4(t_i, t_f, V)} + |F|^2\Big)$$

where D_2 and D_3 depend only in $|\phi|^2_{L^2(t_i, t_f, V)}$, i.e., a bound for $|\phi|^2_{L^2(t_i, t_f, V)}$ induces a bound for both D_2 and D_3.

For the continuity: consider two solutions

$$y_t = -\nu A(y+\phi) - B(y+\phi) + \nu C(y+\phi) + F$$

and

$$x_t = -\nu A(x+\psi) - B(x+\psi) + \nu C(x+\psi) + F;$$

their difference $\eta = y - x$ satisfies, for $\xi = \phi - \psi$,

$$\eta_t = -\nu A(\eta+\xi) - B(y+\phi) + B(x+\psi) + \nu C(\eta+\xi).$$

Thus, since

$$\begin{aligned}
&\Big(-B(y+\phi) + B(x+\psi),\, \eta\Big)\\
=&b(\eta+\xi,\, \eta+\xi,\, y+\phi) + \Big(B(y+\phi) - B(x+\psi),\, \xi\Big)\\
=&b(\eta+\xi,\, \eta+\xi,\, y+\phi) + b(\eta+\xi,\, y+\phi,\, \xi) + b(x+\psi,\, \eta+\xi,\, \xi)\\
=&b(\eta+\xi,\, \eta+\xi,\, y+\phi) + b(\eta+\xi,\, y+\phi,\, \xi) + b(y+\phi,\, \eta+\xi,\, \xi) - b(\eta+\xi,\, \eta+\xi,\, \xi)\\
\leq&K\|\eta+\xi\|\Big(|\eta+\xi|\|y+\phi\| + \|y+\phi\|\|\xi\| + |\eta+\xi|\|\xi\|\Big)\\
\leq&K_1\|\eta+\xi\|\Big(|\eta|(\|y+\phi\| + \|\xi\|) + \|\xi\|\|y+\phi\| + \|\xi\|^2\Big)
\end{aligned}$$

we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\eta|^2\\
&\leq -\nu\|\eta+\xi\|^2 + \nu\|\eta+\xi\|\|\xi\| + K\|\eta+\xi\||\eta|\\
&\quad + K_1\|\eta+\xi\|\Big(|\eta|(\|y+\phi\| + \|\xi\|) + \|\xi\|\|y+\phi\| + \|\xi\|^2\Big)\\
&\leq -\frac{\nu}{2}\|\eta+\xi\|^2 + D\Big(|\eta|^2(\|y+\phi\|^2 + \|\xi\|^2 + 1) + \|\xi\|^2\|y+\phi\|^2 + \|\xi\|^4 + \|\xi\|^2\Big).
\end{aligned}$$

So

$$\begin{aligned}
|\eta|^2_{C([t_i,\, t_f],\, H)} \leq& D_2\left(|\eta(t_i)|^2 + |\xi|^2_{C([t_i,\, t_f],\, V)}(|y+\phi|^2_{L^2(t_i,\, t_f,\, V)} + 1) + |\xi|^4_{L^4(t_i,\, t_f,\, V)}\right)\\
\leq& D_3\left(|\eta(t_i)|^2 + |\xi|^2_{C([t_i,\, t_f],\, V)} + |\xi|^4_{C([t_i,\, t_f],\, V)}\right)
\end{aligned}$$

and

$$|\eta|^2_{L^2(t_i,\, t_f,\, V)} \leq D_4\left(|\eta(t_i)|^2 + |\xi|^2_{C([t_i,\, t_f],\, V)} + |\xi|^4_{C([t_i,\, t_f],\, V)}\right)$$

where D_3 and D_4 depend only in $|\xi|^2_{L^2(t_i,\, t_f,\, V)}$ and $|y+\phi|^2_{L^2(t_i,\, t_f,\, V)}$. A bound for $|\xi|_{C([t_i,\, t_f],\, V)}$ induces a bound for both D_3 and D_4 (y and ϕ being fixed). □

3.2.5 Proof of statement (3.4)

For the proof we will need the following lemma:

Lemma 3.2.3. *Let $\{z(\cdot,\, \sigma) \in W^{1,2}([t_i,\, t_f],\, \mathbb{R}) \mid \sigma \in \Sigma\}$ be a family uniformly bounded w.r.t. σ, i.e., there exists $C > 0$ such that*

$$|z(\cdot,\, \sigma)|_{C([t_i,\, t_f],\mathbb{R})} + \left|\frac{d}{dt}z(\cdot,\, \sigma)\right|_{L^2(t_i,\, t_f,\, \mathbb{R})} \leq C; \quad \forall \sigma \in \Sigma.$$

Then there exists a constant D_1 depending only on C and on $(t_f - t_i)$ such that

$$|\sin(wt)z(t,\, \sigma)|_{rx} \leq D_1 w^{-1}, \quad \text{and} \quad |\cos(wt)z(t,\, \sigma)|_{rx} \leq D_1 w^{-1}.$$

Moreover bounds for C and for the length $(t_f - t_i)$ induces a bound for D_1.

Proof. The proof follows by direct computation: integrating by parts we obtain

$$\left|\int_r^s \sin(wt)z(t,\,\sigma)\,dt\right| \le w^{-1}|z(\cdot,\,\sigma)|_{C([t_i,\,t_f])} + w^{-1}\left|\frac{d}{dt}z(\cdot,\,\sigma)\right|_{L^1(t_i,\,t_f)}$$
$$\le w^{-1}C(1+(t_f-t_i)^{1/2}).$$

Similarly for $|\int_r^s \cos(wt)z(t,\,\sigma)\,dt|$. □

Corollary 3.2.4. *Let $\{z(\cdot,\,\sigma)\in W^{1,2}([0,\,T],\,\mathbb{R})\mid \sigma\in\Sigma\}$ be a family uniformly bounded w.r.t. σ:*

$$|z(\cdot,\,\sigma)|_{C([t_i,\,t_f],\mathbb{R})} + \left|\frac{d}{dt}z(\cdot,\,\sigma)\right|_{L^2(t_i,\,t_f,\mathbb{R})} \le C;\quad C>0,\,\forall\sigma\in\Sigma.$$

Then there exists a constant D_2 depending only on C such that

$$|\phi^w(\cdot,\,b)z(\cdot,\,\sigma)|_{rx} \le D_2 w^{-1}.$$

Proof. Let s, r be in $[0,\,T]$. Without loss of generality, assume that $s<r$. We set $D:=\{t\in[0,\,T]\mid \phi^w(t,\,b)\ne\sin(wt)\}$. Then

$$\int_s^r \phi^w(t,\,b)z(t,\,\sigma)\,dt = \int_{D\cap[s,\,r]}\phi^w(t,\,b)z(t,\,\sigma)\,dt + \int_{[s,\,r]\setminus D}\sin(wt)z(t,\,\sigma)\,dt$$
$$\le 2\frac{T}{w}C + mw^{-1}C(1+T^{1/2}) \le w^{-1}C(2T+m(1+T^{1/2})).$$

Note that since m is the number of intervals of constancy, the set $[s,\,r]\setminus D$ is a union of at most m intervals. Choose $D_2 = C\big(2T+m+T^{\frac{1}{2}}m\big)$. □

Proof of statement (3.4). Let y and v be, respectively, the solutions of

$$y_t = -\nu A(y+\sqrt{2}\phi^w a_j) - B(y+\sqrt{2}\phi^w a_j) + \nu C(y+\sqrt{2}\phi^w a_j) + F + e_j,$$
$$y(t_{j-1}) = y_{j-1}\in V$$

and

$$v_t = -\nu Av - Bv + \nu Cv + F + e_j - Ba_j,\qquad v(t_{j-1}) = v_{j-1}\in V.$$

For the difference $\eta = v-y$ we find

$$\begin{aligned}\eta_t =& -\nu A\eta + \nu\sqrt{2}\phi^w Aa_j - Bv + By + \nu C\eta - \nu\sqrt{2}\phi^w Ca_j\\ &\sqrt{2}\phi^w\Big(B(y,\,a_j)+B(a_j,\,y)\Big) + (2(\phi^w)^2-1)Ba_j\\ \eta(t_{j-1}) =&\ v_{j-1}-y_{j-1}.\end{aligned}$$

Multiplying by η we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\eta|^2 =& -\nu\|\eta\|^2 + \nu(C\eta,\,\eta) + \nu\sqrt{2}\phi^w(Aa_j - Ca_j,\,\eta) + (By-Bv,\,\eta)\\ &+\sqrt{2}\phi^w\Big(B(y,\,a_j)+B(a_j,\,y),\,\eta\Big) + (2(\phi^w)^2-1)(Ba_j,\,\eta).\end{aligned}\tag{3.6}$$

The term $(By - Bv, \eta)$ is equal to $b(\eta, \eta, v)$ so, bounded by $K|\eta|\|\eta\|\|v\|$ and; by the boundedness of $\|v\|$ we have $(By - Bv, \eta) \le \frac{\nu}{2}\|\eta\|^2 + D|\eta|^2$. For $\nu(C\eta, \eta)$ we also have a bound $\nu K\|\eta\||\eta| \le \frac{\nu}{2}\|\eta\|^2 + D_1|\eta|^2$. Therefore from (3.6) we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\eta|^2 \le & D_2|\eta|^2 + \nu\sqrt{2}\phi^w(Aa_j - Ca_j, \eta)\\ & + \sqrt{2}\phi^w\Big(B(y, a_j) + B(a_j, y), \eta\Big) + (2(\phi^w)^2 - 1)(Ba_j, \eta)\end{aligned}$$

and, by the Gronwall inequality, for $t_{j-1} \le s \le t_j$

$$\begin{aligned}|\eta(s)|^2 \le & e^{2D_2(s-t_{j-1})}|\eta(t_{j-1})|^2 + 2\int_{t_{j-1}}^{s} \nu\sqrt{2}\phi^w\Big(Aa_j - Ca_j, \eta(t)\Big)\exp\Big(2D_2(s-t)\Big)\,dt\\ & + 2\int_{t_{j-1}}^{s} \sqrt{2}\phi^w\Big(B(y, a_j) + B(a_j, y), \eta\Big)\exp\Big(2D_2(s-t)\Big)\,dt\\ & + 2\int_{t_{j-1}}^{s} (2(\phi^w)^2 - 1)(Ba_j, \eta)\exp\Big(2D_2(s-t)\Big)\,dt. \end{aligned}\tag{3.7}$$

Now we claim that the scalar product

$$(Aa_j - Ca_j + B(y, a_j) + B(a_j, y), \eta(t))(b)$$

is uniformly bounded in $W^{1,2}(0, T, \mathbb{R})$, with respect to the parameter $b \in B$. Indeed from the uniform boundedness of the family of controls $\{e_j(b) - Ba_j(b) \mid b \in B\}$ in $L^2(0, T, H)$ we derive the uniform boundedness of the family of solutions $v(b)$ in $C(0, T, V)$, $L^2(0, T, D(A))$ and $W^{1,2}(0, T, H)$. Similarly from the uniform boundedness of the family $\{\sqrt{2}\phi^w a_j(b) \mid b \in B\}$ in $L^4(0, T, D(A))$ we derive the uniform boundedness of the family of solutions $y(b)$ in $C(0, T, V)$, $L^2(0, T, D(A))$ and $W^{1,2}(0, T, H)$. Therefore also the family of differences $\eta(b)$ is uniformly bounded in $C(0, T, V)$, $L^2(0, T, D(A))$ and $W^{1,2}(0, T, H)$. Then from the estimates

$$\begin{aligned} &|(Aa_j, \eta)| \le K|a_j|_{[2]}|\eta|; && |b(a_j, y, \eta)| \le K|a_j|_{[2]}\|y\||\eta|;\\ &|(Ca_j, \eta)| \le K\|a_j\||\eta|; && |b(y, a_j, \eta)| \le K|a_j|_{[2]}\|y\||\eta|;\end{aligned}$$

and

$$\begin{aligned} &|(Aa_j, \eta_t)| \le K|a_j|_{[2]}|\eta_t|; && |b(a_j, y_t, \eta) + b(a_j, y, \eta_t)| \le K|a_j|_{[2]}(|y_t|\|\eta\| + \|y\||\eta_t|);\\ &|(Ca_j, \eta_t)| \le K\|a_j\||\eta_t|; && |b(y_t, a_j, \eta) + b(y, a_j, \eta_t)| \le K|a_j|_{[2]}(|y_t|\|\eta\| + \|y\||\eta_t|);\end{aligned}$$

we conclude the uniform boundedness of

$$(Aa_j - Ca_j + B(y, a_j) + B(a_j, y), \eta)(b)$$

in $C([0, T], \mathbb{R})$ and $W^{1,2}(0, T, \mathbb{R})$.

Since $t \mapsto \exp\Big(2D_2(s-t)\Big)$ is uniformly bounded by $K\exp\Big(2D_2T\Big)$ in both $C([0, T], \mathbb{R})$ and $W^{1,2}(0, T, \mathbb{R})$ we have that

$$(Aa_j - Ca_j + B(y(t), a_j) + B(a_j, y(t)), \eta(t))\exp\Big(2D_2(s-t)\Big)$$

is uniformly bounded in $W^{1,2}(0,\,T,\,\mathbb{R})$: note that for $f,\,g$ in $W^{1,2}(0,\,T,\,\mathbb{R})$

$$\begin{aligned}
&|fg|_{W^{1,2}(0,\,T,\,\mathbb{R})}\\
\leq&|fg|_{L^2(0,\,T,\,\mathbb{R})}+|f_tg|_{L^2(0,\,T,\,\mathbb{R})}+|fg_t|_{L^2(0,\,T,\,\mathbb{R})}\\
\leq&K(|f|_{C([0,\,T],\,\mathbb{R})}+|g|_{C([0,\,T],\,\mathbb{R})})(|f_t|_{L^2(0,\,T,\,\mathbb{R})}+|g|_{L^2(0,\,T,\,\mathbb{R})}+|g_t|_{L^2(0,\,T,\,\mathbb{R})}).
\end{aligned}$$

By (3.7) and corollary 3.2.4

$$|\eta(s)|^2\leq e^{2D_2(s-t_{j-1})}|\eta(t_{j-1})|^2+D_3w^{-1}+2\int_{t_{j-1}}^{s}(2(\phi^w)^2-1)(Ba_j,\,\eta)\exp\Big(2D_2(s-t)\Big)\,dt. \tag{3.8}$$

We write the last term as

$$2\int_I(2(\phi^w)^2-1)(Ba_j,\,\eta)\exp\Big(2D_2(s-t)\Big)\,dt \\ +2\int_{[t_{j-1},\,s]\setminus I}(2(\phi^w)^2-1)(Ba_j,\,\eta)\exp\Big(2D_2(s-t)\Big)\,dt$$

where

$$I=[t_{j-1},\,s]\cap[t_{i-1}+\frac{L_j}{w},\,t_j-\frac{L_j}{w}],\qquad L_j=t_j-t_{j-1}.$$

For the last integral we have

$$\int_{[t_{j-1},\,s]\setminus I}(2(\phi^w)^2-1)(Ba_j,\,\eta)\exp\Big(2D_2(s-t)\Big)\,dt\leq K_1\frac{2L_j}{w}$$

and for the first one

$$\int_I(2(\phi^w)^2-1)(Ba_j,\,\eta)\exp\Big(2D_2(s-t)\Big)\,dt \\ =\int_I(-\cos(2wx))(Ba_j,\,\eta)\exp\Big(2D_2(s-t)\Big)\,dt\leq K_2w^{-1},$$

using corollary 3.2.3.

Therefore from (3.8)

$$|\eta(s)|^2\leq D|\eta(t_{j-1})|^2+Dw^{-1}; \tag{3.9}$$

So for small $|\eta(t_{j-1})|$ and big w we have small $|\eta(s)|$. This ends the proof of equation (3.4). $\square$

3.3 Controllability on observed component

Definition 3.3.1. *Let $\phi^0:\,M^1\to M^2$ be a continuous map between two finite dimensional C^0-manifolds, $B\subset M^1$ be an open subset with compact closure and, $S\subseteq M^2$ be any subset. We say that $\phi^0(B)$* **covers** *S* **solidly***, if for any ϕ in some C^0-neighborhood $\mathcal{N}$ of $\phi^0\,|_{\overline{B}}$ there holds: $S\subseteq\phi(B)$.*

Let $\mathcal{O}\subset H$ be a finite dimensional subspace we want to observe. Let P^O be the orthogonal projection map from H onto $\mathcal{O}$. Define, for each $T>0$ and each finite dimensional subspace $\mathbb{F}\subset H$, the "end point" map

$$\mathbb{E}_T:\,V\times L^\infty([0,\,T],\,\mathbb{F})\to\mathcal{O} \\ (u_0,\,v)\mapsto P^O\circ\mathbb{S}_s(u_0,\,F,\,v,\,\nu)(T),$$

where $\mathbb{S}_s(u_0,\, F,\, v,\, \nu)$ is the strong solution of the system

$$u_t = -\nu Au - Bu + \nu Cu + F + v; \qquad u(0) = u_0. \tag{3.10}$$

Definition 3.3.2. *We say that system* (3.10) *is* **time-T solidly controllable on observed component** *if for any $u_0 \in V$, $R > 0$ and any finite dimensional subspace $\mathcal{O} \subset H$, there exists a family of controls*

$$\mathcal{V}_{u_0,R} := \{v_b \in L^\infty([0,\, T], \mathbb{F}) \mid b \in B_{u_0,R}\}$$

such that $\mathbb{E}_T(u_0,\, B_{u_0,R}) := \mathbb{E}_T(u_0,\, \mathcal{V}_{u_0,R})$ covers $\overline{\mathcal{O}}_R(P^O u_0)$ solidly (we consider $\mathbb{E}_T$ as a map from $B_{u_0,R}$ to $\mathcal{O}$: $\mathbb{E}_T(u_0,\, b) := \mathbb{E}_T(u_0,\, v_b)$). $B_{u_0,R}$ is an open relatively compact subset of a C^0-manifold and; $\overline{\mathcal{O}}_R(y)$ is the closed ball

$$\{x \in \mathcal{O} \mid |x - y| \leq R\}.$$

The respective open balls will be denoted by $\mathcal{O}_R(y)$. Note that the family $\mathcal{V}_{u_0,\,R}$ does depend on u_0, R and $\mathcal{O}$.

Proposition 3.3.1. *Let $g \subset V$ be a V-saturating set. Then the system*

$$u_t = -\nu Au - Bu + \nu Cu + F + v; \qquad u(0) = u_0; \qquad v \in G^0 := span\{g\}, \tag{3.11}$$

is time-T solidly controllable on observed component.

Before the proof, for any $N \in \mathbb{N}$, we define the system

$$[N] : \begin{cases} u_t & = -\nu Au - Bu + \nu Cu + F + v, \quad v \in G^N; \\ u(0) & = u_0. \end{cases} \tag{3.12}$$

and we have

Proposition 3.3.2.

1. *For some $T^0 > 0$, every $0 < T \leq T^0$ and every $N \in \mathbb{N}_0$ the system $[(3.12).N]$ is time-T solid controllable in observed component;*

2. *For each pair $(u_0,\, R) \in V \times [0,\, +\infty[$ the family*

$$\mathcal{V}_{u_0,R} := \{v_b \mid b \in B_{u_0,R}\}$$

can be chosen satisfying:

- *The map $b \mapsto v_b$ is $(B,\, L^2(0,\, T,\, G^N))$-continuous and;*
- *The controls $v_b(t)$ are uniformly l_1-bounded w.r.t. b and t:*

$$|v_b(t)| \leq A = A(T, R, u_0).$$

Proposition 3.3.2 is proven in two steps:

First step: The proposition holds for big enough N.
First suppose the finite-dimensional space $\mathcal{O}$ is a subspace of V and fix $\gamma > 1$. The family of constant controls

$$\mathcal{V} := \{v_p := pT^{-1} \mid p \in \mathcal{O}_{\gamma R}(0)\}$$

satisfy the second point of the proposition 3.3.2 and; $\mathbb{E}_T(u_0,\, \mathcal{O}_{\gamma R}(0))$ covers $\overline{\mathcal{O}}_R(P^O u_0)$ solidly, where $\mathbb{E}_T(u_0,\, p)$ is the end point map

$$\mathbb{E}_T(u_0,\, p) := P^O \circ \mathbb{S}_s(u_0,\, F,\, v_p,\, \nu)(T).$$

Indeed consider the solution of the system

$$u_t = -\nu Au - Bu + \nu Cu + F + v_p, \qquad u(0) = u_0 \quad t \in [0,\, T];$$

re-scaling time $t = \xi T$:

$$u_\xi = T(-\nu Au - Bu + \nu Cu + F) + p, \qquad u(0) = u_0 \quad \xi \in [0,\, 1].$$

Let $y = y_0 + p\xi$. For the difference $z = u - y$ we obtain

$$z_\xi = T(-\nu Au - Bu + \nu Cu + F); \qquad z(0) = z_0 = u_0 - y_0$$

and

$$\begin{aligned}\frac{1}{2T}\frac{d}{dt}|z|^2 =& (-\nu Au - Bu + \nu Cu + F,\, z)\\ \leq& -\nu\|z\|^2 + \nu\|y\|\|z\| + \nu K(\|y\| + \|z\|)|z| + |F||z| + |(Bu,\, z)|.\end{aligned}$$

Since

$$\begin{aligned}b(u,\, u,\, z) &= b(y+z,\, y+z,\, z) = b(y+z,\, y,\, z)\\ &= b(y,\, y,\, z) + b(z,\, y,\, z)\end{aligned}$$

we have that $|(Bu,\, z)| \leq K\|y\|^2\|z\| + K|z|\|z\|\|y\|$ and

$$\frac{1}{2T}\frac{d}{dt}|z|^2 \leq C(\|y\|^2 + \|y\|^4 + |F|^2) + C|z|^2(1 + \|y\|^2).$$

Therefore, by Gronwall inequality,

$$\begin{aligned}|z(s)|^2 &\leq \exp\left\{TC\int_0^1 1 + \|y(\xi)\|^2\, d\xi\right\}\left(|z(0)|^2 + TC\int_0^1 \|y(\xi)\|^2 + \|y(\xi)\|^4 + |F|^2\, d\xi\right)\\ &\leq \exp(T)D_1(|z(0)|^2 + TD_2)\end{aligned}$$

where D_1 and D_2 depend only on γ, R and $\|y_0\|$. Indeed $y(\xi) = y_0 + p\xi$ and $\|y_0 + p\xi\| \leq \|y_0\| + C\|p\xi\|_{l_1}$ and, $\|p\xi\|_{l_1} < \gamma R$. In particular we have that

Corollary 3.3.3. *If $y_0 = u_0$, then $|u - y| \leq [T\exp(T)]^{\frac{1}{2}}K$; with K independent of T (K depends only on γ, R and $\|u_0\|$). Moreover for u_0, and R satisfying $\|u_0\| \leq \bar{\mu}$, $R \leq \bar{\rho}$ (and for fixed $\gamma > 1$), we have that K may be taken independent of u_0 and R.*

Now, let $\{e_i \mid i = 1, \ldots, r\} \subset V$ be a basis for $\mathcal{O}$ and let δ be a small positive real number. Set $N \in \mathbb{N}$ such that for each element e_i of this basis we have $|e_i - P^N e_i| < \delta$ and; for each $p = \sum_{i=1}^r p_i e_i$ put $\tilde{p} = \sum_{i=1}^r p_i P^N e_i$ and $v_{\tilde{p}} := \tilde{p}T^{-1}$. Here $P^N : H \to G^N$ denotes the orthogonal projection from H onto G^N.

For small δ we have that the controls $v_{\tilde{p}}$ and v_p are close in H so, at final time T, also $\mathbb{S}_s(u_0, F, v_{\tilde{p}}, \nu)(T)$ and $\mathbb{S}_s(u_0, F, v_p, \nu)(T)$ are close in H. Then for any $p \in \mathcal{O}_{\gamma R}(0)$ we have that $\mathbb{E}(u_0, \tilde{p})$ is close to $\mathbb{E}(u_0, p)$. Hence, for small δ and small T, $\mathbb{E}(u_0, \tilde{p})$ is close to $y_0 + p$, by a degree theory argument, $\mathbb{E}(u_0, \tilde{p})$ covers $\overline{\mathcal{O}}_R(P^O u_0)$ solidly.

Now we observe that for a given finite-dimensional subspace $\widehat{\mathcal{O}} \subset H$, with $\{f_1, \ldots, f_r\}$ being an orthonormal (in H) basis for $\widehat{\mathcal{O}}$ and; for given $u_0 \in V$, $\gamma > 1$, $R > 0$ and for small $\zeta > 0$; we may find an orthonormal (in H) basis $\{e_1, \ldots, e_r\}$ spanning a subspace $\mathcal{O} \subset V$ with $|f_i - e_i| < \zeta$ for all $i = 1, \ldots, r$.

From above we have that there exists of a family $\tilde{p}T^{-1}$ of controls taking values in some space G^N, for big enough N and small enough T, such that $\mathbb{S}_s(u_0, F, v_{\tilde{p}}, \nu)(T)$ is close to $u_0 + p$ in H where p runs over the elements of $\mathcal{O}$ with $|p| < \gamma R$. Therefore $P^{\widehat{O}}\mathbb{S}_s(u_0, F, v_{\tilde{p}}, \nu)(T)$ is close to $P^{\widehat{O}}u_0 + P^{\widehat{O}}p$.

Writing $p = \sum_{i=1}^r p_i e_i$ and defining $\hat{p} = \sum_{i=1}^r p_i f_i$ we have that for small enough $\zeta > 0$, $\hat{p}$ is close to p in H and, $P^{\widehat{O}}\mathbb{S}_s(u_0, F, v_{\tilde{p}}, \nu)(T)$ is close to $P^{\widehat{O}}u_0 + P^{\widehat{O}}\hat{p} = P^{\widehat{O}}u_0 + \hat{p}$. Again by a degree theory argument, we may derive that $P^{\widehat{O}}\mathbb{S}_s(u_0, F, v_{\tilde{p}}, \nu)(T)$ covers $\overline{\widehat{\mathcal{O}}}_R(P^{\widehat{O}}u_0)$ solidly. Therefore we may conclude that, for small enough $T^0 > 0$ and big enough $N \in \mathbb{N}$, the system [(3.12).N] is time-T solidly controllable on observed component for any $T \leq T^0$.

Remark 3.3.1. *For Degree Theory we may refer for example to [25]. The argument we use is simple: if a continuous function f cover a open set containing a compact set, then any other continuous function close enough to f in C^0 cover that compact subset. See [48, subsection 4.10.1] for details.*

Second step: If the proposition holds for $N \geq 1$, then it holds for $N - 1$. Suppose that proposition 3.3.2 holds for a given N. We are given a family of controls $\gamma(t, b)$ taking values on G^N and satisfying the proposition. As we have seen in subsection 3.2.2, we may replace this family by a family of controls $\tilde{z}_{N-1}(t, b)$ taking values on G^{N-1} and leading to close points at time T.

This ends the proof of proposition 3.3.2.

Proof of proposition 3.3.1. If $T \leq T^0$, then proposition 3.3.1 "is contained' in proposition 3.3.2 taking $N = 0$.

If $T > T^0$ we proceed as follows: Fix a finite-dimensional space $\mathcal{O} \subset H$ we want to observe. Applying zero control for time T, we now that the solution of the equation satisfies $\|S_s(u_0, F, 0, \nu)(\xi)\| \leq \|u_0\| + L$ for some positive constant $L > 0$ and all $\xi \in [0, T]$; by corollary 3.3.3, and by the above discussion, we may see that we may set small enough $T_1 < T^0$ and a family of controls $c_{v_0}(\cdot, b)$ taking values in G^0, such that $b \mapsto P^O S_s(v_0, F, c_{v_0}(\cdot, b), \nu)(T_1)$ covers solidly the ball $\mathcal{O}_{R+L+2|u_0|}(P^O v_0)$; for all v_0 satisfying $\|v_0\| \leq \bar{\mu} = \|u_0\| + L$.

Therefore, starting from point u_0, we may apply zero control for time $T - T_1$ which will lead us to the point $v_0 := S_s(u_0, F, 0, \nu)(T - T_1)$. Then we may find a family of controls $c_{v_0}(\cdot, b)$ taking values in G^0, such that $b \mapsto P^O S_s(v_0, F, c_{v_0}(\cdot, b), \nu)(T_1)$ covers solidly the ball $\mathcal{O}_{R+L+2|u_0|}(P^O v_0) \supset \mathcal{O}_R(P^O u_0)$.

The wanted family of controls is given by $v(t,\, b) := \begin{cases} 0 & \text{if} \quad t \in [0,\, T - T_1[\\ c_{v_0}(\cdot,\, b) & \text{if} \quad t \in [T - T_1,\, T] \end{cases}$. □

3.4 H-approximate controllability

We have that for any $T > 0$, system (3.11) is **time-T approximately controllable** in H-norm, i.e.,

Proposition 3.4.1. *Let $g \subset V$ be a V-saturating set. Then for any $u_0 \in V$ and $T > 0$, the attainable set at time T, from u_0, of system* (3.11) *is dense in H.*

Proof. We want to drive the system from u_0 to some neighborhood of u_1. By the density of V in H we may suppose $u_1 \in V$. First put $\|u_0 - u_1\| =: a$ and let $b > 0$ satisfy $\|S_s(u_0,\, F,\, 0,\, \nu)(\xi)\| \leq \|u_0\| + b$ for all $\xi \in [0,\, T]$.

Set $T_1 < T$. Applying zero control for time $T - T_1$, we arrive to the point $v_0 = S_s(v_0,\, F,\, 0,\, \nu)(T - T_1)$. Set N big enough such that both $|v_0 - P^N v_0|$ and $|u_1 - P^N u_1|$ are small. By corollary 3.3.3, if T_1 is small enough, the control $(P^N u_1 - P^N v_0)T_1^{-1}$ drives the system in time T_1 to a point close to $v_0 + (P^N u_1 - P^N v_0)$, i.e., close to u_1. Note that, by corollary 3.3.3, we may choose the time T_1 being small enough for all v_0 satisfying $\|v_0\| \leq \|u_0\| + b$ and all $R \leq a + b + 2|u_0|$; note also that $\|P^N u_1 - P^N v_0\| \leq a + b + 2\|u_0\|$, i.e., we have a bound for $p = \|P^N u_1 - P^N v_0\|$ independent of v_0 and of N.

Finally, we may imitate the dynamics given by the control

$$v(t) := \begin{cases} 0 & \text{if} \quad t \in [0,\, T - T_1[\\ (P^N u_1 - P^N v_0)T_1^{-1} & \text{if} \quad t \in [T - T_1,\, T] \end{cases}$$

taking values in G^N, by the dynamics of a control taking values in G^0 in such a way that at final time T the two dynamics are close. □

Chapter 4

Euclidean domains

In this chapter we consider the case of a two-dimensional plane domain.

Under classical boundary conditions, such as no-slip or Navier, we write the Navier-Stokes equation

$$u_t + (u \cdot \nabla)u + \nabla p = -\nu \Delta u + F(x);$$
$$\nabla \cdot u = 0 \quad \text{in} \quad \Omega;$$

governing a fluid in a bounded domain $\Omega \subseteq \mathbb{R}^2$, as an evolutionary equation

$$u_t = -\nu Au - Bu + \nu Cu + F;$$

where the operators A, B and C have the desired properties (ch. 1) to conclude that the existence of a V-saturating set (ch. 2) is sufficient for controllability results (ch. 3).

We recall that **Navier** boundary conditions read

$$u \cdot \mathbf{n} = 0 \quad \text{on} \quad \partial\Omega; \tag{4.1}$$
$$\nabla^\perp \cdot u = \beta u \cdot \mathbf{t} \quad \text{on} \quad \partial\Omega; \tag{4.2}$$

where β is a function defined on the boundary $\partial\Omega$ of Ω while, **no-slip** boundary conditions read

$$u = 0 \quad \text{on} \quad \partial\Omega. \tag{4.3}$$

Above $\Delta = -\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}$ is the Laplace-de Rham operator; $\nabla \cdot u = -\frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2}$ is the divergence of the vector field $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$; $\nabla^\perp \cdot u = -\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}$ is the vorticity of u; $\nabla p = \begin{pmatrix} \frac{\partial p}{\partial x_1} \\ \frac{\partial p}{\partial x_2} \end{pmatrix}$ is the gradient of the scalar function p and; $\mathbf{n}$ and $\mathbf{t}$ are respectively the normal and tangent vector fields to the boundary $\partial\Omega$ of Ω.

4.1 Preliminaries

Definition 4.1.1. *An open and connected subset $\tilde{\Omega}$ in $\mathbb{R}^N$ ($N \in \mathbb{N}_0$), is called a* **domain** *in $\mathbb{R}^N$.*

Definition 4.1.2. *Let $\tilde{\Omega}$ be a bounded set in $\mathbb{R}^N$, $N \in \mathbb{N}_0$ and $k \in \mathbb{N}$. We say that $\tilde{\Omega}$ is of class $\mathfrak{R}^{k,1}$ if locally (up to a change of coordinates), its boundary Γ is the graph of a C^k function f with $D^\alpha f$ Lipschitz for all $|\alpha| = k$ and; locally $\tilde{\Omega}$ is located on one side of Γ.*

We recall also the definitions of C^k and Lipschitz sets in $\mathbb{R}^N$, $k \in \mathbb{N} \cup \{+\infty\}$:

Definition 4.1.3. *Let $\tilde{\Omega}$ be a set in $\mathbb{R}^N$. We say that $\tilde{\Omega}$ is of class* $\boldsymbol{C^k}$ *(resp.* **Lipschitz***) if locally, its boundary Γ is the graph of a C^k function (resp. a continuous Lipschitz function) and; locally $\tilde{\Omega}$ is located on one side of Γ.*

In particular a bounded Lipschitz set is a $\mathfrak{R}^{0,1}$ set and; a $\mathfrak{R}^{k,1}$ set is a bounded C^k set.

4.1.1 Recollection of auxiliary material on Sobolev spaces H^m

Let us fix a plane bounded domain $\Omega \subset \mathbb{R}^2$ and put $\Gamma := \partial\Omega$. We assume that

$$\left.\begin{array}{l} \bullet\, \Omega \text{ is of class } C^3 \text{ and;} \\ \bullet\, \Gamma \text{has a finite number of connected components denoted} \\ \quad \Gamma_1,\, \Gamma_2,\, \cdots,\, \Gamma_k \quad (k \geq 1). \end{array}\right\} \tag{4.4}$$

Recall that since $L^2(\Omega)$ is a Hilbert space for the scalar product

$$(u,\, v) := (u,\, v)_0 := \int_\Omega uv\, dx, \tag{4.5}$$

then $L^2(T\Omega) = (L^2(\Omega))^2$ is a Hilbert space for the product topology; the scalar product is

$$(u,\, v) := \int_\Omega u \cdot v\, dx. \quad ^1 \tag{4.6}$$

We note that for $u,\, v \in L^2(T\Omega)$

$$(u,\, v) = \int_\Omega u \cdot v\, dx = \sum_{i=1}^{2} \int_\Omega u_i v_i\, dx = (u_1,\, v_1) + (u_2,\, v_2).$$

The norms associated with the previous scalar products shall be represented by

$$|u| := (u,\, u)^{\frac{1}{2}} \tag{4.7}$$

and, for vectors we have

$$|u|^2 := |u_1|^2 + |u_2|^2.$$

Similarly, the Sobolev space $H^1(\Omega)$ is a Hilbert space for the scalar product

$$(u,\, v)_1 := (u,\, v) + \sum_{i=1}^{2} \left(\frac{\partial u}{\partial x_i},\, \frac{\partial v}{\partial x_i} \right) = (u,\, v) + (\nabla u,\, \nabla v). \tag{4.8}$$

then $H^1(T\Omega)$ is a Hilbert space for the product topology; the scalar product is

$$(u,\, v)_1 := \sum_{j=1}^{2} (u_j,\, v_j)_1 = (u,\, v) + \sum_{i=1}^{2} (\nabla u_i,\, \nabla v_i). \tag{4.9}$$

[1] We will use the same notation for the scalar products and norms in $L^2(\Omega)$ and $L^2(T\Omega)$. It will be clear, in the statements, when functions are real or vector so, no ambiguity will appear. For the same reason below we use the same notation for the usual scalar products and norms of $H^m(\Omega)$ and $H^m(T\Omega)$, $\quad m \geq 1$.

The norms associated with the previous scalar products shall be represented by

$$|u|_1 := (u,\, u)_1^{\frac{1}{2}}. \tag{4.10}$$

For vectors we have

$$|u|_1^2 := |u_1|_1^2 + |u_2|_1^2.$$

Similarly we denote the usual scalar product in $H^m(\Omega)$ by

$$(u,\, v)_m := (u,\, v)_{m-1} + \sum_{|\alpha|=m} \left(\frac{\partial^{|\alpha|} u}{\partial x^\alpha},\, \frac{\partial^{|\alpha|} v}{\partial x^\alpha} \right). \tag{4.11}$$

where, as usual, $\alpha = (\alpha_1,\, \alpha_2) \in \mathbb{N}^2$, $|\alpha| = \alpha_1 + \alpha_2$ and ∂x^α stays for $\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}$. Then $H^m(T\Omega)$ is a Hilbert space for the product topology; the scalar product is

$$(u,\, v)_m := \sum_{j=1}^{2} (u_j,\, v_j)_m. \tag{4.12}$$

The norms associated with the previous scalar products shall be represented by

$$|u|_m := (u,\, u)_m^{\frac{1}{2}}. \tag{4.13}$$

Again, for vectors

$$|u|_m^2 := |u_1|_m^2 + |u_2|_m^2.$$

4.1.2 Characterization of $H^m(T\Omega)$

In [[56], Appendix I] we find the following:

Proposition 4.1.1. *Assume that Ω satisfies* (4.4) *and also that Ω is of class C^r with $r \geq m+1$. Then*

$$H^m(T\Omega) = \left\{ u \in L^2(T\Omega) \mid \nabla \cdot u \in H^{m-1}(\Omega), \nabla^\perp \cdot u \in H^{m-1}(\Omega),\, u \cdot \mathbf{n} \in H^{m-\frac{1}{2}}(\Gamma) \right\}$$

and, there exists a constant $C_0 = C_0(m,\, \Omega)$ such that

$$|u|_m \leq C_0 \Big(|u| + |\nabla \cdot u|_{m-1} + |\nabla^\perp \cdot u|_{m-1} + |u \cdot \mathbf{n}|_{H^{m-\frac{1}{2}}(\Gamma)} \Big)$$

for every $u \in H^m(T\Omega)$.

In particular, for our domain satisfying (4.4) we have:

Corollary 4.1.2. *There is a constant C_0 such that*

$$|u|_1 \leq C_0 \Big(|u| + |\nabla \cdot u| + |\nabla^\perp \cdot u| + |u \cdot \mathbf{n}|_{H^{1-\frac{1}{2}}(\Gamma)} \Big)$$

for every $u \in H^1(T\Omega)$ and;

$$|u|_2 \leq C_0 \Big(|u| + |\nabla \cdot u|_1 + |\nabla^\perp \cdot u|_1 + |u \cdot \mathbf{n}|_{H^{2-\frac{1}{2}}(\Gamma)} \Big)$$

for every $u \in H^2(T\Omega)$.

We have the following trace theorem (see [40] section 2.5.4):

Lemma 4.1.3. *Let $\tilde{\Omega} \subseteq \mathbb{R}^N$ be a $\mathfrak{R}^{k-1,1}$ open set, $p > 1$, $k \in \mathbb{N}_0$, $u \in W^{k,p}(\tilde{\Omega})$. Then for $l \leq k-1$ there holds:*

$$\left|\frac{\partial^l u}{\partial u^l}\right|_{W^{k-l-\frac{1}{p}}(\partial\tilde{\Omega})} \leq C|u|_{W^{k,p}(\tilde{\Omega})}.$$

In particular we have that

Corollary 4.1.4. *For our fixed domain $\Omega \subset \mathbb{R}^2$, there exists a constant C_0 such that*

$$|u|_{H^{1-\frac{1}{2}}(\Gamma)} \leq C_0|u|_1 \quad \textit{and} \quad |v|_{H^{2-\frac{1}{2}}(\Gamma)} \leq C_0|v|_2.$$

every $u \in H^1(\Omega)$, $v \in H^2(\Omega)$.

Corollary 4.1.5. *The norms*

$$|u|_1 \textit{ and } \left(|u|^2 + |\nabla \cdot u|^2 + |\nabla^\perp \cdot u|^2 + |u \cdot \mathbf{n}|^2_{H^{1-\frac{1}{2}}}\right)^{\frac{1}{2}}$$

are equivalent norms in $H^1(T\Omega)$ and, the norms

$$|u|_2 \textit{ and } \left(|u|^2 + |\nabla \cdot u|_1^2 + |\nabla^\perp \cdot u|_1^2 + |u \cdot \mathbf{n}|^2_{H^{2-\frac{1}{2}}}\right)^{\frac{1}{2}}$$

are equivalent norms in $H^2(T\Omega)$.

Remark 4.1.1. *In the simply-connected case [looking at [56], Appendix I, Equation (1.28)] we have the inequality*

$$|u|_m \leq C_1\Big(|\nabla \cdot u|_{m-1} + |\nabla^\perp \cdot u|_{m-1} + |u \cdot \mathbf{n}|_{H^{m-\frac{1}{2}}(\Gamma)}\Big).$$

and then we conclude the equivalence of the norms

$$|u|_1 \textit{ and } \left(|\nabla \cdot u|^2 + |\nabla^\perp \cdot u|^2 + |u \cdot \mathbf{n}|^2_{H^{1-\frac{1}{2}}}\right)^{\frac{1}{2}}$$

in $H^1(T\Omega)$ and, of the norms

$$|u|_2 \textit{ and } \left(|\nabla \cdot u|_1^2 + |\nabla^\perp \cdot u|_1^2 + |u \cdot \mathbf{n}|^2_{H^{2-\frac{1}{2}}}\right)^{\frac{1}{2}}$$

in $H^2(T\Omega)$.

4.2 Navier boundary conditions

4.2.1 The spaces and the linear operator A

We set

$$\begin{aligned} H &:= \{u \in L^2(T\Omega) \mid \nabla \cdot u = 0 \;\;\&\;\; u \cdot \mathbf{n} = 0 \,\text{on}\, \Gamma\}; \\ \text{and} & \\ V &:= \{u \in H^1(T\Omega) \mid \nabla \cdot u = 0 \;\;\&\;\; u \cdot \mathbf{n} = 0 \,\text{on}\, \Gamma\}. \end{aligned} \tag{4.14}$$

In H we consider the scalar product induced by $L^2(T\Omega)$ and respective norm.

We suppose the function β, in (4.2), to be of class $C^1(\Gamma)$ so that, we may extend it to a $C^1(\tilde{\Omega})$ function defined in a neighborhood $\tilde{\Omega}$ of $\overline{\Omega}$. We may also extend the normal $\mathbf{n}$ to a $C^2(\tilde{\Omega})$ function. We fix one extension of β and one of $\mathbf{n}$ that we will denote again by β and $\mathbf{n}$. Note that in this way the "tangent" $\mathbf{t} := \begin{pmatrix} -\mathbf{n}_2 \\ \mathbf{n}_1 \end{pmatrix} \in C^2(\tilde{\Omega})$ is an extension of the tangent $\mathbf{t} \in C^2(\Gamma)$.

Define on V the bilinear form

$$((u,\, v)) := (\nabla^\perp \cdot u,\, \nabla^\perp \cdot v) + D_0(u,\, v) - (\beta u \cdot \mathbf{t},\, \nabla^\perp \cdot v) - (\beta v \cdot \mathbf{t},\, \nabla^\perp \cdot u)$$

and let D_0 satisfy

$$\left(D_0 - \frac{1}{2}\right) |u|^2 \geq 2|\beta u \cdot \mathbf{t}|^2, \quad \text{for all} \quad u \in H. \quad ^2 \tag{4.15}$$

The symmetry and bilinearity of $((\cdot,\, \cdot))$ is clear. After a little computation for $((u,\, u))$ we find

$$\frac{1}{2}\left(|\nabla^\perp \cdot u|^2 + |u|^2\right) \leq ((u,\, u)) \leq (2D_0 + 1)\left(|\nabla^\perp \cdot u|^2 + |u|^2\right) \tag{4.16}$$

from which, using corollary 4.1.5, we conclude that $((u,\, u))$ is a scalar product on V and, its associated norm

$$\|\cdot\| := ((\cdot,\, \cdot))^{\frac{1}{2}}$$

is equivalent to the norm induced in V by the usual norm of $H^1(T\Omega)$ defined in (4.10).

From now we consider V endowed with the scalar product $((\cdot,\, \cdot))$ and respective norm. Since H and V are closed subspaces of $L^2(T\Omega)$ and $H^1(T\Omega)$ respectively, they are Hilbert spaces.

We denote by A the canonical isomorphism between V and V' associated to $((\cdot,\, \cdot))$, i.e., $A : V \to V'$

$$((u,\, v)) =: \langle Au,\, v\rangle_{V',V}.$$

The inclusions (identifying H with its dual)

$$V \subset H \subset V'$$

are both continuous and dense. For $v \in V$ and $u \in H$ we have $\langle u,\, v\rangle_{V',V} = (u, v)$.
The domain $D(A)$ of the operator A in H is defined as

$$D(A) := \{u \in V \mid Au \in H\};$$

A is a strictly positive unbounded linear operator in H with domain $D(A)$.

We endow $D(A)$ with the scalar product $(u,\, v)_{[2]} := (Au,\, Av)$ and respective norm $|u|_{[2]} = |Au|$.

From the compactness of the injection $V \to H$ follow the compactness of the operator A^{-1}.

---

[2] Note that $2|\beta u \cdot \mathbf{t}|^2 \leq C|u|^2$ where C depends only in the $C^0(\Omega)$-norm of the previously fixed functions β and $\mathbf{t}$. Then set $D_0 = C + \frac{1}{2}$.

Characterization of $D(A)$

We prove the characterization:

$$D(A) = D_A := \{u \in H^2(T\Omega) \mid \nabla \cdot u = 0;\ (\nabla^\perp \cdot u = \beta u \cdot \mathbf{t} \wedge u \cdot \mathbf{n} = 0) \text{ on } \Gamma\} \tag{4.17}$$

and the equivalence of the norms $|u|_{[2]} := |Au|$ and $|u|_2$ on $D(A)$:

Define the operator

$$\begin{aligned} L : V &\to H \\ u &\mapsto Lu := P^\nabla[(\nabla^\perp \cdot u)\beta \mathbf{t}]. \end{aligned}$$

So $(Lu,\, v) = ((\nabla^\perp \cdot u)\beta\mathbf{t},\, v) = (\nabla^\perp \cdot u,\, \beta v \cdot \mathbf{t})$, for all $v \in H$. Note that $v \mapsto (Lu,\, v)$ is linear and continuous as v varies in H.

For every test function

$$\varphi \in (\mathcal{D}(\Omega))^2;\ \mathcal{D}(\Omega) := \{u \in C^\infty(\Omega) \mid \mathit{supp}(u) \subset \Omega\}, \qquad ^3$$

we write $\varphi = P^\nabla \varphi + \nabla\psi$, where P^∇ stays for the orthogonal projection from $L^2(T\Omega)$ onto H and, $\nabla\phi$ belongs to the space

$$H^\perp = \{\nabla u \mid\ u \in H^1(\Omega)\} \tag{4.18}$$

orthogonal to H (in $L^2(T\Omega)$).

It is known (see for example the proof of theorem 1.5 in [56, section I.1.4]) that ϕ is the solution of the Neumann problem

$$\begin{aligned} \Delta\phi &= \nabla \cdot \varphi \\ \frac{\partial\phi}{\partial\mathbf{n}} &= \varphi \cdot \mathbf{n}; \end{aligned}$$

thus for $P^\nabla\varphi$ we have:

$$\begin{aligned} &P^\nabla\varphi \in L^2(T\Omega); \qquad \nabla^\perp \cdot P^\nabla\varphi = \nabla^\perp \cdot \varphi \in L^2(\Omega); \\ &\nabla \cdot P^\nabla\varphi = \nabla \cdot \varphi - \Delta\phi = 0 \in L^2(\Omega); \qquad P^\nabla\varphi \cdot \mathbf{n} = \varphi \cdot \mathbf{n} - \frac{\partial\phi}{\partial\mathbf{n}} = 0 \in H^{1-\frac{1}{2}}(\Gamma); \end{aligned}$$

from which, using proposition 4.1.1, we obtain $P^\nabla\varphi \in V$.

The curl $\nabla^\perp f$ of the scalar function f is defined as $\nabla^\perp f = \begin{pmatrix} -\frac{\partial f}{\partial x_2} \\ \frac{\partial f}{\partial x_1} \end{pmatrix}$; for $u \in D(A)$ we have

$$\begin{aligned} &\langle \Delta u + D_0 u + \nabla^\perp(\beta u \cdot \mathbf{t}) - Lu,\, \varphi\rangle \\ =&(\nabla^\perp \cdot u,\, \nabla^\perp \cdot \varphi) + D_0(u,\, \varphi) - (\beta u \cdot \mathbf{t},\, \nabla^\perp \cdot \varphi) + (Lu,\, \varphi) \\ =&(\nabla^\perp \cdot u,\, \nabla^\perp \cdot P^\nabla\varphi) + D_0(u,\, P^\nabla\varphi) - (\beta u \cdot \mathbf{t},\, \nabla^\perp \cdot P^\nabla\varphi) + (Lu,\, P^\nabla\varphi) \\ =&(Au,\, P^\nabla\varphi) = (Au,\, \varphi); \end{aligned} \tag{4.19}$$

[3]Here $\mathit{supp}(u)$ stays for the support of u defined by: $\mathit{supp}(u) :=$ closure of $\{x \in \Omega \mid\ u(x) \neq 0\}$.

where $\langle \cdot, \cdot \rangle$ stays for the scalar product in the duality between $((\mathcal{D}(\Omega))^2)' = (\mathcal{D}'(\Omega))^2$ and $(\mathcal{D}(\Omega))^2$ and; $(\cdot, \cdot)$ stays for the scalar product in $L^2(T\Omega)$. Note that $\Delta u = \nabla(\nabla \cdot u) - \nabla^\perp(\nabla^\perp \cdot u) = -\nabla^\perp(\nabla^\perp \cdot u)$ because, $\nabla \cdot u = 0$ for $u \in V$.

Therefore we conclude that $\Delta u + D_0 u + \nabla^\perp(\beta u \cdot \mathbf{t}) - Lu$ and $Au \in H \subset L^2(T\Omega)$ are the same distribution in $(\mathcal{D}'(\Omega))^2$:

$$\Delta u + D_0 u + \nabla^\perp(\beta u \cdot \mathbf{t}) - Lu = Au \in L^2(T\Omega); \quad u \in D(A). \tag{4.20}$$

From $\Delta u + D_0 u + \nabla^\perp(\beta u \cdot \mathbf{t}) - Lu \in L^2(T\Omega)$ and $u \in D(A) \subset V$ we obtain:

$$\begin{aligned} &u \in L^2(T\Omega); \quad \nabla^\perp \cdot u \in L^2(\Omega); \quad \nabla \cdot u = 0 \in H^1(\Omega); \\ &(u \cdot \mathbf{n}) \mid_\Gamma = 0 \in H^{2-\frac{1}{2}}(\Gamma); \quad \nabla(\nabla^\perp \cdot u) = \begin{pmatrix} -\Delta u_2 \\ \Delta u_1 \end{pmatrix} \in L^2(T\Omega). \end{aligned}$$ [4]

Hence from proposition 4.1.1 we have $u \in H^2(T\Omega)$. In particular $\nabla^\perp \cdot u \in H^1(\Omega)$ and so, the trace $(\nabla^\perp \cdot u)\mathbf{t}$ belong to $(H^{1-\frac{1}{2}}(\Gamma))^2$. [5]

Now, for $v \in H^1(T\Omega)$, considering that $P^\nabla v$ and $\nabla\phi$ are the orthogonal projections of v onto H and $H^\perp$ respectively; the Green formula gives

$$\begin{aligned} &(\Delta u + D_0 u + \nabla^\perp(\beta u \cdot \mathbf{t}) - Lu, \, v) \\ =&(\nabla^\perp \cdot u, \, \nabla^\perp \cdot v) + D_0(u, \, v) - (\beta u \cdot \mathbf{t}, \, \nabla^\perp \cdot v) - (Lu, \, v) \\ &\qquad + \int_\Omega \nabla^\perp \cdot ((-\nabla^\perp \cdot u + \beta u \cdot \mathbf{t})v) \, dx \\ =&(\nabla^\perp \cdot u, \, \nabla^\perp \cdot P^\nabla v) + D_0(u, \, P^\nabla v) - (\beta u \cdot \mathbf{t}, \, \nabla^\perp \cdot P^\nabla v) - (Lu, \, P^\nabla v) \\ &\qquad + \int_\Gamma (-\nabla^\perp \cdot u + \beta u \cdot \mathbf{t}) v \cdot \mathbf{t} \, d\Gamma \\ =&(Au, \, P^\nabla v) + \int_\Gamma (-\nabla^\perp \cdot u + \beta u \cdot \mathbf{t}) v \cdot \mathbf{t} \, d\Gamma. \end{aligned} \tag{4.21}$$

Note that since ϕ is the solution of the Neumann problem

$$\begin{aligned} \Delta\phi &= \nabla \cdot v \\ \frac{\partial \phi}{\partial \mathbf{n}} &= v \cdot \mathbf{n}; \end{aligned}$$

we have:

$$\begin{aligned} &P^\nabla v \in L^2(T\Omega); \qquad \nabla^\perp \cdot P^\nabla v = \nabla^\perp \cdot v \in L^2(\Omega); \\ &\nabla \cdot P^\nabla v = \nabla \cdot v - \Delta\phi = 0 \in L^2(\Omega); \qquad P^\nabla v \cdot \mathbf{n} = v \cdot \mathbf{n} - \frac{\partial \phi}{\partial \mathbf{n}} = 0 \in H^{1-\frac{1}{2}}(\Gamma); \end{aligned}$$

from which, using proposition 4.1.1, we obtain $P^\nabla v \in V$.
Since

$$(Au, \, v) = (Au, \, P^\nabla v)$$

[4]From $\Delta u + D_0 u + \nabla^\perp(\beta u \cdot \mathbf{t}) - Lu \in L^2(T\Omega)$, with $u \in H^1(T\Omega)$, we obtain $\begin{pmatrix} \Delta u_1 \\ \Delta u_2 \end{pmatrix} \in L^2(T\Omega)$.

[5]Note that $\mathbf{t} \in (C^2(\Gamma))$

by (4.20) and (4.21) we conclude that

$$\int_\Gamma (\nabla^\perp \cdot u - \beta u \cdot \mathbf{t}) v \cdot \mathbf{t}\, d\Gamma = 0, \qquad \forall v \in H^1(T\Omega),$$

in particular for $v = (\nabla^\perp \cdot u - \beta u \cdot \mathbf{t})\mathbf{t}$ we obtain $(\nabla^\perp \cdot u - \beta u \cdot \mathbf{t})^2 = 0$ on Γ.

Up to now we have concluded that

$$D(A) \subseteq D_A;$$

next we prove the reverse inclusion and so we have the characterization (4.17) for $D(A)$.

First we note that for $b \in D_A$ and for $v \in V$ we find

$$(\Delta b + D_0 b + \nabla^\perp(\beta b \cdot \mathbf{t}) - Lb,\, v) = \langle Ab,\, v\rangle_{V',V}$$

then, to prove that $D(A) \supseteq D_A$, is enough to prove that $\Delta b + D_0 b + \nabla^\perp(\beta b \cdot \mathbf{t}) - Lb \in H$ because, in that case we have necessarily $\Delta b + D_0 b + \nabla^\perp(\beta b \cdot \mathbf{t}) - Lb = Ab$.

On Ω we have

$$\nabla \cdot (\Delta b + D_0 b + \nabla^\perp(\beta b \cdot \mathbf{t}) - Lb) = 0$$

and, on Γ we have

$$\mathbf{n} \cdot (\Delta b + D_0 b + \nabla^\perp(\beta b \cdot \mathbf{t}) - Lb) = (\mathbf{n} \cdot \nabla^\perp)(-\nabla^\perp \cdot u + \beta u \cdot \mathbf{t}) = 0$$

because, since $-\nabla^\perp \cdot u + \beta u \cdot \mathbf{t}$ is constant on Γ, we have that $\nabla^\perp(-\nabla^\perp \cdot u + \beta u \cdot \mathbf{t})$ is tangent to Γ. [6]

Therefore $\Delta b + D_0 b + \nabla^\perp(\beta b \cdot \mathbf{t}) - Lb \in H$ and, then $b \in D(A)$.

Remark 4.2.1. *Defining*

$$\mathcal{D}_1(\Omega) := \{\varphi \in C^\infty(\overline{\Omega}) \mid \nabla \cdot \varphi = 0,\, (\varphi \cdot \mathbf{n} = 0 \wedge \nabla^\perp \cdot \varphi = \beta u \cdot \mathbf{t}\; on\, \Gamma)\};$$

we have the following characterizations:

$$\begin{aligned} H &= \textit{ closure of} \mathcal{D}_1(\Omega)\, in\, L^2(T\Omega); \\ V &= \textit{ closure of} \mathcal{D}_1(\Omega)\, in\, H^1(T\Omega); \\ D(A) &= \textit{ closure of} \mathcal{D}_1(\Omega)\, in\, H^2(T\Omega). \end{aligned}$$

Indeed it is known that H is the closure of $\mathcal{V} := \{\varphi \in \mathcal{D}(\Omega) \mid \nabla \cdot \varphi = 0\}$ in $L^2(T\Omega)$ [7] and, from $\mathcal{V} \subset \mathcal{D}_1(\Omega) \subset H$, follows that H is the closure of $\mathcal{D}_1(\Omega)$ in $L^2(T\Omega)$. It is also clear that $D(A)$ is the closure of $\mathcal{D}_1(\Omega)$ in $H^2(\Omega)$ and then, by the density, and continuity of the inclusion, of $D(A)$ into V we can conclude the density of $\mathcal{D}_1(\Omega)$ in V.

[6] Indeed it is well known that ∇g is normal to the curve γ if g is constant on γ; on the other side $\nabla^\perp g$ is orthogonal to ∇g.

[7] See [56], section 1.1.4.

Now, for $u \in D(A)$, we compare the norms $|u|_{[2]} = |Au|$ and $|u|_2$:

$$\begin{aligned}
(Au,\, Au) &= (\Delta u + D_0 u + \nabla^\perp(\beta u \cdot \mathbf{t}) - Lu,\, Au)\\
&= (\Delta u,\, Au) + (D_0 u,\, Au) + (\nabla^\perp(\beta u \cdot \mathbf{t}) - Lu,\, Au)\\
&= |\Delta u|^2 + (\Delta u,\, D_0 u) + (\Delta u,\, \nabla^\perp(\beta u \cdot \mathbf{t})) - (\Delta u,\, Lu) + D_0\|u\|^2\\
&\quad + (\nabla^\perp(\beta u \cdot \mathbf{t}),\, \Delta u) - (Lu,\, \Delta u) + D_0(\nabla^\perp(\beta u \cdot \mathbf{t}),\, u) - D_0(Lu,\, u)\\
&\quad + |\nabla^\perp(\beta u \cdot \mathbf{t}) - Lu|^2\\
&= |\Delta u|^2 + D_0\|u\|^2 + |\nabla^\perp(\beta u \cdot \mathbf{t}) - Lu|^2 + 2(\Delta u,\, \nabla^\perp(\beta u \cdot \mathbf{t}) - Lu)\\
&\quad + D_0[(\Delta u,\, u) + (\nabla^\perp(\beta u \cdot \mathbf{t}),\, u) - (Lu,\, u)]\\
&= |\Delta u|^2 + D_0\|u\|^2 + |\nabla^\perp(\beta u \cdot \mathbf{t}) - Lu|^2 + 2(\Delta u,\, \nabla^\perp(\beta u \cdot \mathbf{t}) - Lu)\\
&\quad + D_0\Big[|\nabla^\perp \cdot u|^2 - \int_\Gamma (\nabla^\perp \cdot u) u \cdot \mathbf{t}\, d\Gamma - (\nabla^\perp \cdot u,\, \beta u \cdot \mathbf{t})\\
&\quad + \int_\Gamma (\beta u \cdot \mathbf{t}) u \cdot \mathbf{t}\, d\Gamma - (Lu,\, u)\Big];
\end{aligned}$$

i.e.,

$$\begin{aligned}
|Au|^2 = |\Delta u|^2 &+ D_0\|u\|^2 + |\nabla^\perp(\beta u \cdot \mathbf{t}) - Lu|^2 + D_0|\nabla^\perp \cdot u|^2\\
&+ 2(\Delta u,\, \nabla^\perp(\beta u \cdot \mathbf{t}) - Lu) - 2D_0(Lu,\, u).
\end{aligned} \tag{4.22}$$

From

$$\Big|2(\Delta u,\, \nabla^\perp(\beta u \cdot \mathbf{t}) - Lu)\Big| \le 2|\Delta u||\nabla^\perp(\beta u \cdot \mathbf{t}) - Lu| \le \frac{1}{2}|\Delta u|^2 + 2|\nabla^\perp(\beta u \cdot \mathbf{t}) - Lu|^2$$

we obtain

$$\begin{aligned}
|Au|^2 \ge \frac{1}{2}&|\Delta u|^2 + D_0\|u\|^2 + D_0|\nabla^\perp \cdot u|^2\\
&- |\nabla^\perp(\beta u \cdot \mathbf{t}) - Lu|^2 - 2D_0(Lu,\, u).
\end{aligned} \tag{4.23}$$

Now for D_0 big enough, namely if D_0 satisfies both (4.15) and

$$\frac{D_0}{2}|u|^2 \ge \frac{9}{4}|\beta u \cdot \mathbf{t}|^2, \quad \text{for all} \quad u \in V \tag{4.24}$$

we have

$$\begin{aligned}
\frac{D_0}{2}\|u\|^2 + \frac{D_0}{2}|\nabla^\perp \cdot u|^2 =& \frac{D_0}{2}\Big(|\nabla^\perp \cdot u|^2 + D_0|u|^2 - 2(Lu,\, u)\Big) + \frac{D_0}{2}|\nabla^\perp \cdot u|^2\\
=& D_0\Big(|\nabla^\perp \cdot u|^2 + \frac{D_0}{2}|u|^2 - (Lu,\, u)\Big)\\
\ge& D_0\Big(|\nabla^\perp \cdot u|^2 + \frac{9}{4}|\beta u \cdot \mathbf{t}|^2 - (Lu,\, u)\Big)
\end{aligned}$$

and, since

$$3(Lu,\, u) = 3(\beta u \cdot \mathbf{t},\, \nabla^\perp \cdot u) \le |\nabla^\perp \cdot u|^2 + \frac{9}{4}|\beta u \cdot \mathbf{t}|^2$$

we have

$$\frac{D_0}{2}\|u\|^2 + \frac{D_0}{2}|\nabla^\perp \cdot u|^2 \ge 2D_0(Lu,\, u).$$

Hence from (4.23) we obtain

$$|Au|^2 \geq \frac{1}{2}|\Delta u|^2 + \frac{D_0}{2}\|u\|^2 + \frac{D_0}{2}|\nabla^\perp \cdot u|^2 - |\nabla^\perp(\beta u \cdot \mathbf{t}) - Lu|^2. \tag{4.25}$$

We know that the norm $\|\cdot\|$ is equivalent to norm induced by the usual norm $|\cdot|_1$ of $H^1(T\Omega)$ in V. Then there exists a constant C_1 such that, for all $u \in H^1(T\Omega)$ holds $|\nabla^\perp(\beta u\cdot\mathbf{t}) - Lu| \leq C_1|u|_1$; we may choose D_0 satisfying

$$\frac{D_0-1}{2}\|u\|^2 \geq |\nabla^\perp(\beta u \cdot \mathbf{t}) - Lu|^2, \quad \text{for all} \quad u \in V. \tag{4.26}$$

From now on we consider D_0 satisfying all the conditions (4.15), (4.24) and (4.26), i.e.,

$$\left.\begin{array}{r} (D_0 - \frac{1}{2})|u|^2 \geq 2|\beta v \cdot \mathbf{t}|^2, \quad \text{for all} \quad u \in H \\ \frac{D_0}{2}|u|^2 \geq \frac{9}{4}|\beta u \cdot \mathbf{t}|^2 \quad \text{for all} \quad u \in V \\ \frac{D_0-1}{2}\|u\|^2 \geq |\nabla^\perp(\beta u \cdot \mathbf{t}) - Lu|^2 \quad \text{for all} \quad u \in V \end{array}\right\} \tag{4.27}$$

For such a choice, from (4.25) and (4.16), we obtain

$$\begin{aligned} |Au|^2 &\geq \frac{1}{2}|\Delta u|^2 + \frac{1}{2}\|u\|^2 + \frac{D_0}{2}|\nabla^\perp \cdot u|^2 \\ &\geq \frac{1}{2}|\Delta u|^2 + \frac{1}{2}\|u\|^2 \geq \frac{1}{2}|\Delta u|^2 + \frac{1}{4}(|\nabla^\perp \cdot u|^2 + |u|^2) \\ &\geq \frac{1}{4}\Big(|\Delta u|^2 + |\nabla^\perp \cdot u|^2 + |u|^2\Big). \end{aligned} \tag{4.28}$$

On the other hand, by (4.22), we have

$$\begin{aligned} |Au|^2 &\leq |\Delta u|^2 + D_0\|u\|^2 + |\nabla^\perp(\beta u \cdot \mathbf{t}) - Lu|^2 + D_0|\nabla^\perp \cdot u|^2 \\ &\quad + 2|\Delta u||\nabla^\perp(\beta u \cdot \mathbf{t}) - Lu| + 2D_0|(Lu, u)| \\ &\leq |\Delta u|^2 + D_0\|u\|^2 + |\nabla^\perp(\beta u \cdot \mathbf{t}) - Lu|^2 + D_0|\nabla^\perp \cdot u|^2 \\ &\quad + |\Delta u|^2 + |\nabla^\perp(\beta u \cdot \mathbf{t}) - Lu|^2 + D_0|\nabla^\perp \cdot u|^2 + D_0|\beta u \cdot \mathbf{t}|^2 \\ &\leq 2|\Delta u|^2 + D_0\|u\|^2 + 2|\nabla^\perp(\beta u \cdot \mathbf{t}) - Lu|^2 + 2D_0|\nabla^\perp \cdot u|^2 + D_0|\beta u \cdot \mathbf{t}|^2 \\ &\leq 2|\Delta u|^2 + (2D_0 - 1)\|u\|^2 + 2D_0|\nabla^\perp \cdot u|^2 + \frac{D_0}{2}\left(D_0 - \frac{1}{2}\right)|u|^2 \\ &\leq 2|\Delta u|^2 + (2D_0 - 1)(2D_0 + 1)(|\nabla^\perp \cdot u|^2 + |u|^2) \\ &\quad + 2D_0|\nabla^\perp \cdot u|^2 + \frac{D_0}{2}\left(D_0 - \frac{1}{2}\right)|u|^2 \end{aligned}$$

and since

$$2 + (2D_0 - 1)(2D_0 + 1) + 2D_0 + \frac{D_0}{2}\left(D_0 - \frac{1}{2}\right) \leq 1 + 4D_0^2 + 2D_0 + D_0^2 \leq 5(D_0 + 1)^2$$

we obtain

$$|Au|^2 \leq 5(D_0 + 1)^2\big(|\Delta u|^2 + |\nabla^\perp \cdot u|^2 + |u|^2\big). \tag{4.29}$$

Hence from (4.28), (4.29) and corollary 4.1.5 we have that the norm $|\cdot|_{[2]} := |Au|$ is equivalent to the norm induced by the usual norm $|\cdot|_2$ in $D(A)$. Note that $|\Delta u| = |\nabla\nabla^\perp \cdot u|$ for each divergence free vector field $u \in H^2(T\Omega)$.

4.2.2 The linear operator C

We set

$$Cu := P^{\nabla}[D_0 u + \nabla^{\perp}(\beta u \cdot \mathbf{t}) - Lu] = D_0 u + P^{\nabla}[\nabla^{\perp}(\beta u \cdot \mathbf{t})] - Lu;$$

the operator C from V to H is symmetric in $V \times V$, when seen as $C(u, v) := (Cu, v)$, and satisfies

$$(Cu, v) \leq K\|u\||v|; \quad u \in V, \ v \in H.$$

The symmetry follows from

$$\begin{aligned}(Cu, v) &= (P^{\nabla}[D_0 u + \nabla^{\perp}(\beta u \cdot \mathbf{t}) - Lu], v) = (D_0 u + \nabla^{\perp}(\beta u \cdot \mathbf{t}) - Lu, v) \\ &= D_0(u, v) - (\beta u \cdot \mathbf{t}, \nabla^{\perp} \cdot v) - (\nabla^{\perp} \cdot u, \beta v \cdot \mathbf{t}) + \int_{\Gamma} \beta u \cdot \mathbf{t} v \cdot \mathbf{t} \, d\Gamma\end{aligned}$$

and, the estimate is easy because

$$|D_0 u + \nabla^{\perp}(\beta u \cdot \mathbf{t}) - Lu| \leq D_0|u| + K_1|u|_1 + K_2|\nabla^{\perp} \cdot u| \leq K\|u\|.$$

4.3 No-slip boundary conditions

For the case of no-slip boundary conditions, well studied in [56](see also [24]), the respective subspaces and operator A are

$$\begin{aligned} H &:= \{u \in L^2(T\Omega) : \nabla \cdot u = 0 \ \& \ u \cdot \mathbf{n} = 0 \, \mathrm{on} \, \Gamma\}; \\ V &:= \{u \in H^1(T\Omega) : \nabla \cdot u = 0 \ \& \ u = 0 \, \mathrm{on} \, \Gamma\}; \\ & A : V \to V' \\ & \langle Au, v \rangle := (\nabla u, \nabla v) = (\nabla^{\perp} \cdot u, \nabla^{\perp} \cdot u), \quad u, v \in V; \\ D(A) &:= \{u \in V \mid Au \in H\} = H^2(T\Omega) \cap V. \end{aligned}$$

The bilinear form

$$((u, v)) :=< Au, v >, \quad u, v \in V,$$

is a scalar product on V and its associated norm $\|u\| := ((u, u))^{\frac{1}{2}}$ is equivalent to the norm induced on V by the usual norm of $H^1(T\Omega)$.
The operator $A : D(A) \to H$ is called the Stokes operator and the norm of $D(A)$ defined by $|u|_{[2]} := |Au|$ is equivalent to the norm induced in $D(A)$ by the usual norm of $H^2(T\Omega)$.

For the linear operator C we set $C \equiv 0$.

4.4 The Operator B

We define the trilinear form b by

$$(u, v, w) \mapsto \sum_{i,j=1}^{2} \int_R u_i(\partial_i v_j) w_j \, dx \tag{4.30}$$

for which we have the estimates

$$|b(u, v, w)| \leq C_1 K$$

where C_1 is a constant and K is one of the following products

$$
\begin{aligned}
&|u|_1|v|_1|w|_1 && \text{for } u,\, v,\, w \in H^1(T\Omega);\\
&|u|_0^{\frac{1}{2}}|u|_1^{\frac{1}{2}}|v|_1^{\frac{1}{2}}|v|_2^{\frac{1}{2}}|w|_0 && \text{for } u \in H^1(T\Omega),\, v \in H^2(T\Omega),\, w \in L^2(T\Omega);\\
&|u|_0^{\frac{1}{2}}|u|_2^{\frac{1}{2}}|v|_1|w|_0 && \text{for } u \in H^2(T\Omega),\, v \in H^1(T\Omega),\, w \in L^2(T\Omega);\\
&|u|_0|v|_1|w|_0^{\frac{1}{2}}|w|_2^{\frac{1}{2}} && \text{for } u \in L^2(T\Omega),\, v \in H^1(T\Omega),\, w \in H^2(T\Omega);\\
&|u|_0^{\frac{1}{2}}|u|_1^{\frac{1}{2}}|v|_1|w|_0^{\frac{1}{2}}|w|_1^{\frac{1}{2}} && \text{for } u,\, v,\, w \in H^1(T\Omega).
\end{aligned}
$$

This estimates can be found in [56, section III.3.2]. Mainly they follow by results on interpolation ([35]), generalized Sobolev inequalities ([41]) and by a S. Agmon's theorem ([1, lemma 13.2]). We refer to [54] for indications how to obtain them.

For either Navier or no-slip boundary conditions, by the equivalence of the norms $\|\cdot\|$ and $|\cdot|_1$ in V and of the norms $|\cdot|_{[2]}$ and $|\cdot|_2$ in $D(A)$, we may replace $|\cdot|_1$ by $\|\cdot\|$ and $|\cdot|_2$ by $|\cdot|_{[2]}$ in the estimates above (as soon we have vectors in the spaces H, V or $D(A)$, accordingly with the respective estimate; note that $|\cdot| = |\cdot|_0$). Therefore we have the desired estimates for the trilinear form b.[8]

For pairs $(u,\, v)$ such that $w \mapsto b(u,\, v,\, w)$ is continuous in V, [9] we may define the the operator $B(u,\, v) \in V'$ by

$$
w \mapsto \langle B(u,\, v),\, w\rangle_{V',V} := b(u,\, v,\, w); \tag{4.31}
$$

and denote $B(u) := B(u,\, u)$.

Lemma 4.4.1. *Fixing the first variable in H, the form b results skew-symmetric in the last two variables:*

$$
\begin{aligned}
b(u,\, v,\, w) = -b(u,\, w,\, v)&\\
&\forall u \in V \forall v,\, w \in H^1(T\Omega);\\
&\forall u \in H \forall (v,\, w) \in H^2(T\Omega) \times H^1(T\Omega) \cup H^1(T\Omega) \times H^2(T\Omega).
\end{aligned}
$$

Proof. For $u \in D(A)$ and $(v,\, w) \in H^2(T\Omega) \times H^2(T\Omega)$ we have

$$
\begin{aligned}
b(u,\, v,\, w) &= \sum_{i,\, j=1}^{2} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j \, d\Omega\\
&= \sum_{i,\, j=1}^{2} \int_\Omega \frac{\partial}{\partial x_i}(u_i v_j w_j)\, d\Omega - \sum_{i,\, j=1}^{2} \int_\Omega \frac{\partial u_i}{\partial x_i} v_j w_j \, d\Omega - \sum_{i,\, j=1}^{2} \int_\Omega u_i v_j \frac{\partial w_j}{\partial x_i}\, d\Omega\\
&= \sum_{j=1}^{2} \int_{\partial\Omega} (u \cdot \mathbf{n}) v_j w_j \, d\partial\Omega - \sum_{j=1}^{2} \int_\Omega (\nabla \cdot u) v_j w_j \, d\Omega - \sum_{i,\, j=1}^{2} \int_\Omega u_i v_j \frac{\partial w_j}{\partial x_i}\, d\Omega\\
&= -b(u\,,w,\, v);
\end{aligned}
$$

the lemma follows by a continuity argument. □

[8] As asked in chapter 1.

[9] For example for $u,\, v \in V$.

Corollary 4.4.2. *Fixing the first variable in H, we have*

$$b(u, v, v) = 0, \quad (u, v, w) \in V \times H^1(T\Omega) \times H^1(T\Omega) \cup H \times H^2(T\Omega) \times H^2(T\Omega)$$

Remark 4.4.1. *In [18] Navier boundary conditions are defined as*

$$u \cdot \mathbf{n} = 0 \quad on \quad \partial\Omega;$$
$$2D(u)\mathbf{n} \cdot \mathbf{t} = -\alpha u \cdot \mathbf{t} \quad on \quad \partial\Omega;$$

where $D(u)$ is the "rate of strain tensor" defined by $D_{ij}(u) := \frac{1}{2}\left(\frac{\partial u_i}{\partial u_j} + \frac{\partial u_j}{\partial u_i}\right)$. Also in [18] it is proven that this condition are equivalent to (4.1)-(4.2) with $\beta = \alpha - 2\kappa$, where κ is the curvature of the boundary $\partial\Omega$.

Again in [18] α is a positive function defined on $\partial\Omega$ so, in this case Lions boundary conditions — $\beta \equiv 0$ — would be a particular case of Navier boundary conditions for boundaries with positive curvature, i.e., for convex domains. Like in [34], we impose no restriction on the sign of α.

With the representation $\beta = \alpha - 2\kappa$, we must ask α to be of class $C^1(\partial\Omega)$ to have the desired regularity $\beta \in C^1(\partial\Omega)$: our domain is of class C^3 so, we have $\kappa \in C^1(\partial\Omega)$.

4.5 The vorticity equation

For a C^∞ plane bounded domain Ω with boundary $\partial\Omega$. Let $p \in \mathbb{N}_0$ be a positive natural number and let $g := \{W_i \mid i = 1, \ldots, p\} \subseteq V \cap (C^\infty(\overline{\Omega}))^2$ be a finite set of steady flows of the Euler system, i.e., $B(W_i) = 0$ for all $i \in \{1, \ldots, p\}$. We remark that when $u \in H^2(T\Omega)$, Bu coincides with the orthogonal projection $P^\nabla[(u \cdot \nabla)u]$ of $(u \cdot \nabla)u$ onto the space H of divergence free vector fields in Ω tangent to the boundary $\partial\Omega$.

Note that given $u, v \in H \cap (C^\infty(\overline{\Omega}))^2$ also $B(u, v) \in H \cap (C^\infty(\overline{\Omega}))^2$: $B(u, v) = (u \cdot \nabla)v - \nabla\phi$ where ϕ solves the system

$$\Delta\phi = \nabla \cdot ((u \cdot \nabla)v) \text{ in } \Omega; \quad \nabla\phi \cdot \mathbf{n} = ((u \cdot \nabla)v) \cdot \mathbf{n} \text{ on } \partial\Omega.$$

It is known that $\phi \in H^1(\Omega)$ so in particular

$$\Delta\phi + \phi = \nabla \cdot ((u \cdot \nabla)v) + \phi \in L^2(\Omega); \quad \nabla\phi \cdot \mathbf{n} = ((u \cdot \nabla)v) \cdot \mathbf{n} \in H^{2-\frac{1}{2}(\partial\Omega)}$$

and, by regularity results on elliptic problems (see [29], theorems 2.4.2.7 and 2.5.1.1), we have that first $\phi \in H^2(\Omega)$; then

$$\Delta\phi + \phi = \nabla \cdot ((u \cdot \nabla)v) + \phi \in H^2(\Omega); \quad \nabla\phi \cdot \mathbf{n} = ((u \cdot \nabla)v) \cdot \mathbf{n} \in H^{2+2-1-\frac{1}{2}(\partial\Omega)}$$

and so $\phi \in H^4(\Omega)$. Analogously, for any $k \geq 1$ and $\phi \in H^{2k}(\Omega)$ we arrive to

$$\Delta\phi + \phi = \nabla \cdot ((u \cdot \nabla)u) + \phi \in H^{2k}(\Omega); \quad \nabla\phi \cdot \mathbf{n} = ((u \cdot \nabla)u) \cdot \mathbf{n} \in H^{2+k-1-\frac{1}{2}(\partial\Omega)}$$

and then $\phi \in H^{2(k+1)}(\Omega)$. By Sobolev embedding theorems we deduce that $\phi \in C^\infty(\overline{\Omega})$, i.e., $Bu \in (C^\infty(\overline{\Omega}))^2$.

Therefore starting with a finite number $g = \{W_i \mid i = 1, \ldots, p\} \subset H$ of smooth steady states for the Euler system, for Navier boundary conditions, the recursive step of the definition of l-saturating set reduces to

$$L^{m+1} := L^m + span\{-B(W_i, v) - B(v, W_i) \mid i = 1, \ldots, p, v \in L^m\}. \tag{4.32}$$

4.5.1 $l^\perp$-saturating sets

Consider the space $\mathcal{S}$ of vector fields in $L^2(T\Omega)$ defined by

$$\mathcal{S} := \{\nabla^\perp \psi \mid \psi \in H^1(\Omega)\}.$$

The space $\mathcal{S}$ and its orthogonal in $L^2(T\Omega)$, given by

$$\mathcal{S}^\perp := \{u \in L^2(T\Omega) \mid \nabla^\perp \cdot u = 0,\, u \cdot \mathbf{t} = 0 \text{ on } \partial\Omega\},$$

are closed in $L^2(T\Omega)$.

The vectors in $H \cap \mathcal{S}$ are solenoidal and may be written as $u = -\nabla^\perp \psi$, the function ψ is unique up to an additive constant and, since u is tangent to the boundary we have necessarily that ψ must be constant at the boundary. The function ψ vanishing at the boundary and satisfying $u = -\nabla^\perp \psi$ is called the **stream function** for the solenoidal vector field u.

We see that the vectors in $H \cap \mathcal{S}$ are those in H that may be (uniquely) recovered by the respective vorticity. The stream function for a vector field $u \in H \cap \mathcal{S}$ solves $\Delta\psi = \nabla^\perp \cdot u$ in Ω and $\psi = 0$ on $\partial\Omega$.

Definition 4.5.1. *Consider a finite set $h = \{v_i \mid i = 1, \ldots, p\} \subset (\nabla^\perp \cdot H_E)$, where H_E is the subset of $H \cap \mathcal{S}$ consisting of smooth steady states of the Euler system. The set h is said* $l^\perp$**-saturating** *if the sequence $(L^{\perp,j})_{j\in\mathbb{N}}$ of finite dimensional subspaces defined recursively by*

1. $L^{\perp,0} := span(h)$;
2. $L^{\perp,m+1} := L^{\perp,m} + span\{\{\Delta^{-1}v_i,\, v\} + \{\Delta^{-1}v,\, v_i\} \mid i = 1, \ldots, p,\, v \in L^{\perp,m}\}$

satisfies,

$$\overline{\bigcup_{i\in\mathbb{N}} L^{\perp,i}} = \overline{\nabla^\perp \cdot (V \cap \mathcal{S})}.$$

where the closure is to be taken in the $L^2(\Omega)$-norm.

Remark 4.5.1. *By $\{f,\, g\}$ we mean the Poisson bracket between the functions f and g, i.e., $\{f,\, g\} = \frac{\partial f}{\partial x_1}\frac{\partial g}{\partial x_2} - \frac{\partial f}{\partial x_2}\frac{\partial g}{\partial x_1}$. We have*

$$\nabla^\perp \cdot \big(-B(W_i,\, V) - B(V,\, W_i)\big) = \{\Delta^{-1}v_i,\, v\} + \{\Delta^{-1}v,\, v_i\}$$

for $\nabla^\perp \cdot W_i = v_i$ and $\nabla^\perp \cdot V = v$.

In the simply-connected case we have $H \subset \mathcal{S}$, so:

Corollary 4.5.1. *Under Navier boundary conditions, if Ω has a smooth boundary and is simply-connected, the existence of a $l^\perp$-saturating set is a sufficient condition for both H-approximate controllability and controllability on finite-dimensional observed component.*

Under Navier boundary conditions, if the domain Ω has a smooth boundary and is simply-connected, a $l^\perp$-saturating set h of scalar fields gives us the l-saturating set of vector fields $(\nabla^\perp\cdot)^{-1}h$.

Remark 4.5.2. *In the multi-connected case we have to take some care, working with the vorticity and in order to "translate" the results for the vector equation, we have to restrict ourselves to vector fields in $H \cap \mathcal{S}$. This means that we have to work on the subspaces $\tilde{H} := H \cap \mathcal{S}$, $\tilde{V} := V \cap \mathcal{S}$ and $D(\tilde{A}) := \{u \in \tilde{V} \mid \tilde{A}u \in \tilde{H}\}$.*

We may proceed as follows: take the scalar product in $\tilde{V}$ induced by that in V, then define $\tilde{A}$ and its domain $D(\tilde{A})$ analogously as we have done before in chapter 1.

For no-slip boundary conditions $D(\tilde{A})$ will coincide with the intersection $\tilde{V} \cap H^2(\Omega)$ and on $D(\tilde{A})$ we will have $\tilde{A}u \equiv \tilde{P}(\Delta u)$ where $\tilde{P}$ stays for the orthogonal projection from $L^2(\Omega)$ onto $\tilde{H}$. We note, anyway, that under no-slip boundary conditions, from a $l^\perp$-saturating set we are not able, in general, to derive a V-saturating set of vector fields.

For Navier boundary conditions $D(\tilde{A})$ will coincide with the intersection $\tilde{V} \cap D(A)$ and on $D(\tilde{A})$ we will have $\tilde{A}u \equiv \Delta u + D_0 u + \nabla^\perp(\beta u \cdot \mathbf{t}) - \tilde{L}b$ where $\tilde{L}u$ is given by

$$\tilde{L}u := \tilde{P}[(\nabla^\perp \cdot u)\beta \mathbf{t}].$$

In particular we still have the same spaces $\tilde{H}$ and $\tilde{V}$ for all Navier boundary conditions.

Of course in this case we have to study the evolution of the equation on $\tilde{H}$ instead of on H.

Remark 4.5.3. *Again in the case Ω is a two dimensional multi-connected domain which boundary has a finite number $p+1$ of connected components — $\Gamma = \cup_{i=0}^{p}\Gamma_i$, a vector field in H can be recovered by its vorticity if p circulations $\int_{\Gamma_i} u \cdot \mathbf{t}\, d\Gamma_i$, $(i = 1, \ldots, p)$, are given (see [38, section 1.2]); we see that for*

- *Navier boundary conditions $u \cdot \mathbf{n} = 0$ & $\nabla^\perp \cdot u = \beta u \cdot \mathbf{t}$ on Γ with β a nonzero constant: circulations are necessarily $\frac{1}{\beta}\int_{\Gamma_i} \nabla^\perp \cdot u\, d\Gamma_i$;*
- *no-slip boundary conditions: circulations are necessarily zero.*

Chapter 5

Riemannian domains

Here we consider the case our domain is a two-dimensional Riemannian manifold.

5.1 The Navier-Stokes equation

Consider a compact two-dimensional smooth Riemannian manifold (Ω, g) with metric tensor $g = g_{ij}dx^i \otimes dx^j$, Ω may have a smooth boundary Γ and if $\Gamma \neq \emptyset$ we suppose that Ω, with its boundary, is included in the interior of a bigger manifold $\tilde{\Omega}$, i.e., $\overline{\Omega} = \Omega \cup \Gamma \subset \tilde{\Omega}$. Similarly to the Euclidean case, the Navier-Stokes equation for the vector field of velocities of the fluid "particles" $u = u(x, t)$, on Ω reads:

$$u_t = -\nu\Delta u - \nabla^1_u u + \nabla p + F + v; \quad \nabla \cdot u = 0.$$

The bilinear term in the equation is the Levi-Civita connection $\nabla^1_u v$, i.e., the unique linear connection that is both torsion-free and metric. In Euclidean case we have $\nabla^1_u v \equiv (u \cdot \nabla)v$. In the equation Δ is the Laplace-de Rham operator; ∇ is the gradient operator and $\nabla\cdot$ is the divergence operator. All these operators are well defined in the context of Riemannian manifolds; for those who are not familiar with these definitions we append some notes at the end of this chapter.

5.2 Levi-Civita connection for tensors

For simplicity from now we will denote $\partial_i := \frac{\partial}{\partial x^i}$. It is well known (see for example [31, section 3.3]) that the Levi-Civita connection gives

$$\nabla^1_{\partial_i}\partial_j = \Gamma^k_{ij}\partial_k$$

where Γ^k_{ij} are the Christoffel symbols $\Gamma^k_{ij} = \frac{g^{kl}}{2}(g_{il,j} + g_{jl,i} - g_{ij,l})$ and $g_{pq,r} := \partial_r q_{pq}$.

Computing, for $v = v^i\partial_i$ and $u = u^i\partial_i$,

$$\nabla^1_v u = v^i\nabla^1_{\partial_i} u = \left(v^i u^j \Gamma^k_{ij} + v^i\partial_i u^k\right)\partial_k$$

we have in particular that

$$\nabla^1\partial_j = \Gamma^k_{ij}dx^i \otimes \partial_k.$$

We extend the Levi-Civita connection ∇^1 to tensors as follows (see [50, section 2.2]):

$$\begin{aligned}
\nabla^1 f =& df, \quad \text{for a function } f;\\
\nabla^1 \partial_j =& \Gamma^k_{ij} dx^i \otimes \partial_k;\\
\nabla^1 dx^i :=& -\Gamma^i_{jk} dx^j \otimes dx^k;\\
\nabla^1 \left(\otimes_{j=1}^p \partial_{i_j}\right) =& \sum_{j=1}^p \partial_{i_1} \otimes \cdots \otimes \nabla^1 \partial_{i_j} \otimes \cdots \otimes \partial_{i_p};\\
\nabla^1 \left(\otimes_{j=1}^p dx^{i_j}\right) =& \sum_{j=1}^p dx^{i_1} \otimes \cdots \otimes \nabla^1 dx^{i_j} \otimes \cdots \otimes dx^{i_p};\\
\nabla^1(\alpha \otimes \beta) =& \nabla^1 \alpha \otimes \beta + \alpha \otimes \nabla^1 \beta.
\end{aligned}$$

5.3 Scalar product on tensors

The metric g defining a scalar product on $T_x\Omega$ induces a scalar product on $T_x^*\Omega$ via $g(\alpha, \beta) := g(\alpha^\flat, \beta^\flat)$. Now we see that it also induces a scalar product on each tensor space $T_x^{p,q}\Omega = (\otimes_{i=1}^p T_x^*\Omega) \otimes (\otimes_{j=1}^q T_x\Omega)$: just put

$$g(A, B) := gA(\tilde{B})$$

where, for a "simple" tensor $A = \otimes_{i=1}^{p+q} A_i \in T_x^{p,q}\Omega$, we define $gA := \otimes_{i=1}^{p+q} gA_i$, i.e., $gA := A_1^\flat \otimes \cdots \otimes A_p^\flat \otimes A_{p+1}^\sharp \otimes \cdots \otimes A_{p+q}^\sharp$, because for a vector v and a 1-form w we have $gv = v^\sharp$ and $gw = w^\flat$; $\tilde{B}$ is the natural ordered pair associated to B in the Cartesian product $\prod_{i=1}^p T_x^*\Omega \times \prod_{j=1}^q T_x\Omega$. The scalar product between simple tensors A, $B \in T_x^{p,q}\Omega$ has the form

$$g(A, B) = \prod_{i=1}^{p+q} g(A_i, B_i). \tag{5.1}$$

From simple tensors we extend $g(\cdot, \cdot)$ to a scalar product in all $T_x^{p,q}\Omega$ by bilinearity.

For functions we define $g(f, h) = fh$, and for the product between functions and tensors we define $g(fA_1, hB_1) := g(f, h)g(A_1, B_1)$.

To verify it is a scalar product it remains to check it is positive definite: it is clearly positive definite for length 0 (functions) and length 1 (vector fields and 1-forms) tensors. Suppose now it is definite positive for length $n-1$ tensors, with $n \geq 2$. For simplicity we consider the sum of two simple tensors, for linear combinations of simple tensors we may proceed analogously. Let T_1, T_2 two tensors of length n; write $T_1 = A \otimes B$, $T_2 = C \otimes D$ where A, C have length 1 and B, D have length $n-1$; we obtain

$$\begin{aligned}
& g(A \otimes B + C \otimes D, A \otimes B + C \otimes D) \qquad (5.2)\\
=& g(A, A)g(B, B) + g(C, C)g(D, D) + 2g(A, C)g(B, D)\\
\geq& g(A, A)g(B, B) + g(C, C)g(D, D) - \frac{1}{2}\Big(g(A, A) + g(C, C)\Big)\Big(g(B, B) + g(D, D)\Big)\\
=& \frac{1}{2}\Big(g(A, A) - g(C, C)\Big)\Big(g(B, B) - g(D, D)\Big). \qquad (5.3)
\end{aligned}$$

In the case $g(A,\,A)g(B,\,B) > g(C,\,C)g(D,\,D)$ we may rewrite $A\otimes B$ as $A'\otimes B' = \frac{1+\delta}{\sqrt{g(A,\,A)}}A\otimes \frac{\sqrt{g(A,\,A)}}{1+\delta}B$ and $C\otimes D$ as $C'\otimes D' = \frac{1}{\sqrt{g(C,C)}}C\otimes\sqrt{g(C,\,C)}D$. Taking small enough $\delta > 0$ we have

$$\begin{aligned} &g(A\otimes B + C\otimes D,\, A\otimes B + C\otimes D) = g(A'\otimes B' + C'\otimes D',\, A'\otimes B' + C'\otimes D') \\ \geq &\frac{1}{2}\Big(g(A',\,A') - g(C',\,C')\Big)\Big(g(B',\,B') - g(D',\,D')\Big) > 0. \end{aligned}$$

In the case $g(A,\,A)g(B,\,B) < g(C,\,C)g(D,\,D)$ we proceed analogously.

It remains to consider the case $g(A,\,A)g(B,\,B) = g(C,\,C)g(D,\,D)$. In this case rewrite $A\otimes B$ as $A'\otimes B' = \frac{1}{\sqrt{g(A,\,A)}}A\otimes\sqrt{g(A,\,A)}B$ and $C\otimes D$ as $C'\otimes D' = \frac{1}{\sqrt{g(C,C)}}C\otimes\sqrt{g(C,\,C)}D$. Then we have $g(A',\,A') = g(C',\,C')$ and $g(B',\,B') = g(D',\,D')$. From

$$\begin{aligned} &g(A\otimes B + C\otimes D,\, A\otimes B + C\otimes D) = g(A'\otimes B' + C'\otimes D',\, A'\otimes B' + C'\otimes D') \\ =&g(A',\,A')g(B',\,B') + g(C',\,C')g(D',\,D') + 2g(A',\,C')g(B',\,D') \\ =&\frac{1}{2}\Big(g(A',\,A') + g(C',\,C')\Big)\Big(g(B',\,B') + g(D',\,D')\Big) + \frac{1}{2}\Big(2g(A',\,C')2g(B',\,D')\Big) \end{aligned}$$

and from $|2g(S,\,R)| < g(S,\,S) + g(R,\,R)$ for nonzero tensors $S \neq \pm R$ of length less than n, we have that

$$g(A'\otimes B' + C'\otimes D',\, A'\otimes B' + C'\otimes D') > 0$$

if $C' \notin \{-A',\,A'\}$ and $D' \notin \{-B',\,B'\}$. On the other side if $C' = \pm A'$ we have

$$\begin{aligned} g(A'\otimes B' + C'\otimes D',\, A'\otimes B' + C'\otimes D') =&g(A'\otimes(B'\pm D'),\, A'\otimes(B'\pm D')) \\ =&g(A',\,A')g(B'\pm D',\, B'\pm D'). \end{aligned}$$

Therefore $g(A'\otimes B' + C'\otimes D',\, A'\otimes B' + C'\otimes D')$ vanishes only if we have simultaneously $A' = \pm C'$ and $B = \mp D'$, i.e., if we have $A\otimes B = A'\otimes B' = -C'\otimes D' = -C\otimes D$.

Remark 5.3.1. *In some references the scalar product on k-forms is defined as* $(\alpha,\,\beta)_{sc} = *(\alpha\wedge *\beta)$, *where* $*$ *is the Hodge map. Consider for simplicity the case of simple k-forms;* $\alpha = \alpha_1\wedge\cdots\wedge\alpha_k$ *and* $\beta = \beta_1\wedge\cdots\wedge\beta_k$. *As we may see in [31, section 2.1] we have* $*(\alpha\wedge *\beta) = det(g(\alpha_i,\,\beta_j)) =: \tilde{g}$. *On the other side for the scalar product* $g(\alpha,\,\beta)$ *in* (5.1) *we find*

$$\begin{aligned} g(\alpha,\,\beta) = g&\left(\sum_\sigma sign(\sigma)\otimes_{i=1}^k \alpha_{\sigma(i)},\, \sum_\rho sign(\rho)\otimes_{j=1}^k \beta_{\rho(j)}\right) \\ &= \sum_{\sigma,\rho} sign(\sigma)sign(\rho)\sum_{i=1}^k g(\alpha_{\sigma(i)},\,\beta_{\rho(i)}) = k!\sum_\rho sign(\rho)\sum_{i=1}^k g(\alpha_i,\,\beta_{\rho(i)}) = k!\tilde{g}. \end{aligned}$$

Therefore the two scalar products $(\cdot,\,\cdot)_{sc}$ *and* $g(\cdot,\,\cdot)$ *differ on k-forms by the factor* $k!$.

5.4 Sobolev spaces

Now we may define Sobolev spaces in any tensor space $T^{p,q}\Omega$; essentially on compact manifolds (with or without boundary) they have the same properties as Sobolev spaces defined in a bounded subset in the Euclidean space: via a partition of unity argument we may reduce the study to a small neighborhood in $\overline{\Omega}$ where, our manifold is like an Euclidean bounded set. Although the definition is clear, it is a bit messy to deal with Sobolev spaces on Riemannian manifolds. Details concerning the case of Sobolev spaces in functions in $T^{0,0}\Omega$ may be found in [12]; for the general case in tensors, we refer to [42]; fractional order Sobolev spaces for functions are defined in [50, section 1.3].

Following [42], we denote by $\Gamma^0(T^{p,q}\Omega)$ the set of the (p,q)-tensor fields with compact support on Ω and by $\overline{\Gamma}(T^{p,q}\Omega)$ the set of those (p,q)-tensor fields B for which $\nabla^i B$ can be extended continuously up to the boundary $\partial\Omega$ of Ω, for all $i \geq 0$; by ∇^i we mean $\nabla^1 \circ \nabla^{i-1}$, for $i \geq 1$ and, by ∇^0 we mean the identity: $\nabla^0 A \equiv A$, for all tensors A.

The Lebesgue space $L^s(T^{p,q}\Omega)$, $1 \leq s < \infty$ is the completion of the set $\Gamma_0(T^{p,q}\Omega)$ in the norm

$$|A|_{L^s(T^{p,q}\Omega)} = \left(\int_\Omega g(A,\, A)^{\frac{s}{2}}\, d\Omega\right)^{\frac{1}{s}}.$$

$L^s(T^{p,q}\Omega)$ is known to include $\overline{\Gamma}(T^{p,q}\Omega)$.

For integer m and $1 \leq s < \infty$, the Sobolev space $H^{m,s}(T^{p,q}\Omega)$ is the completion of the set $\overline{\Gamma}(T^{p,q}\Omega)$ in the norm

$$|A|_{H^{m,s}(T^{p,q}\Omega)} = \left(\sum_{i=0}^{m} |\nabla^i A|^s_{L^s(T^{p+i,q}\Omega)}\, d\Omega\right)^{\frac{1}{s}}.$$

For a tensor field $A \in H^{m,s}(T^{p,q}\Omega)$ with $m \geq 1$ we may consider the trace (restriction) $A \mid \partial\Omega$ of A on the boundary. The space of traces is denoted by $H^{m-\frac{1}{s},s}(T^{p,q}\Omega \mid \partial\Omega)$ and endowed with the norm

$$|A|_{H^{m-\frac{1}{s},s}(T^{p,q}\Omega|\partial\Omega)} = \inf_{\substack{B\in H^{m,s}(T^{p,q}\Omega)\\ B|\partial\Omega = A}} |B|_{H^{m,s}(T^{p,q}\Omega)}.$$

In [12], we see that for functions we have the Sobolev and Rellich-Kondrachov imbedding theorems concerning continuity and compactness of inclusions between Sobolev spaces. Once we have those imbedding theorems for functions we also have them for tensor fields due to local componentwise considerations (see [42]).

From now we will work mainly with the Hilbert spaces $H^{m,2}(T\Omega)$ and $H^{m-\frac{1}{2},2}(T^{p,q}\Omega \mid \partial\Omega)$ we shall denote, for simplicity, by $H^m(T\Omega)$ and $H^{m-\frac{1}{2}}(T^{p,q}\Omega \mid \partial\Omega)$.

We use the compactness of our manifold Ω. Consider a partition of unity $\{(\rho_c,\, \tilde{\Omega}_c) \mid c = 1,\, \ldots,\, C\}$ associated to a finite covering $\{\tilde{\Omega}_c \mid c = 1,\, \ldots,\, C\}$ of $\overline{\Omega} = \Omega \cup \partial\Omega$ by open neighborhoods in a bigger manifold containing $\overline{\Omega}$. In each of those neighborhoods we suppose to have (Riemannian) normal coordinates with center p ($\exp_p(\Omega_c) = U_c$) where U_c is open and bounded[1]. For each small ϵ we may set the neighborhoods U_c so that

$$|g_{ij} - \delta_{ij}| < \epsilon; \qquad |g_{ij,k}| < \epsilon$$

[1] see [31, section 1.4]

on each U_c. The symbol δ_{ij} is the Kronecker symbol taking the value 1 for $i = j$ and 0 otherwise.

For a vector field $u = u^i \partial_i$ we have $\nabla^1 u = (\partial_k u^i + u^r \Gamma^i_{rk}) dx^k \otimes \partial_i$. Write $u = \sum_c \rho_c u$ and compute

$$
\begin{aligned}
|u|^2_{H^1(T\Omega)} &= \sum_c \left(\int_{\Omega_c} g_{ij} \rho_c u^i \rho_c u^j \, d\Omega \right) \\
&+ \sum_c \left(\int_{\Omega_c} g^{ks} g^{it} (\partial_k(\rho_c u^i) + \rho_c u^r \Gamma^i_{rk})(\partial_s(\rho_c u^t) + \rho_c u^l \Gamma^t_{ls}) \, d\Omega \right)
\end{aligned}
$$

where $\Omega_c := \tilde{\Omega}_c \cap \Omega$. The terms $g^{ks} g^{it} \rho_c u^r \Gamma^i_{rk} \rho_c u^l \Gamma^t_{ls}$ admit upper estimate $C_\epsilon (\rho_c u^s)^2$; the terms $g^{ks} g^{it} \partial_k(\rho_c u^i) \partial_s(\rho_c u^t)$, $(1 + C_\epsilon)(\partial_k(\rho_c u^i))^2$ and; the terms $g^{ks} g^{it} \partial_k(\rho_c u^i) \rho_c u^l \Gamma^t_{ls}$, $C_\epsilon \partial_k(\rho_c u^i) \rho_c u^l \leq C_\epsilon [(\partial_k(\rho_c u^i))^2 + (\rho_c u^l)^2]$. Here and in what follows C_ϵ stays for some "constant" that goes to zero when so does ϵ. Thus

$$
\begin{aligned}
|u|^2_{H^1(T\Omega)} &\leq (1 + C_\epsilon) \sum_c \left(\int_{\Omega_c} (\rho_c u^i)^2 \, d\Omega_c + \int_{\Omega_c} (\partial_k(\rho_c u^i))^2 \, d\Omega_c \right) \\
&= (1 + C_\epsilon) \sum_c \left(\int_{U_c} (\rho_c u^i)^2 \sqrt{g} dx^1 \wedge dx^2 + \int_{U_c} (\partial_k(\rho_c u^i))^2 \sqrt{g} \, dx^1 \wedge dx^2 \right) \\
&\leq (1 + C_\epsilon) \sum_c \left(\int_{U_c} (\rho_c u^i)^2 \, dx^1 \wedge dx^2 + \int_{U_c} (\partial_k(\rho_c u^i))^2 \, dx^1 \wedge dx^2 \right).
\end{aligned}
$$

Now we use the results in the Euclidean case and obtain for constants D_c

$$
\begin{aligned}
&|u|_{H^1(T\Omega)} \leq \\
&(1 + C_\epsilon) \sum_c D_c \left(|\rho_c u|_{(L^2(U_c))^2} + |\nabla \cdot (\rho_c u)|_{L^2(U_c)} + |\nabla^\perp \cdot (\rho_c u)|_{L^2(U_c)} + |\rho_c u \cdot \mathbf{n}|_{H^{1-\frac{1}{2}}(\partial U_c)} \right).^2
\end{aligned}
$$

Proceeding analogously, going back to our manifold Ω is not hard to see that

$$
|\rho_c u|_{(L^2(U_c))^2} + |\nabla \cdot (\rho_c u)|_{L^2(U_c)} + |\nabla^\perp \cdot (\rho_c u)|_{L^2(U_c)}
$$

is bounded by $(1 + C_\epsilon) \left(|\rho_c u|_{L^2(T\Omega_c)} + |\nabla \cdot (\rho_c u)|_{L^2(\Omega_c)} + |\nabla^\perp \cdot (\rho_c u)|_{L^2(\Omega_c)} \right) + C_\epsilon |\rho_c u|^2_{H^1(T\Omega)}$ while on the boundary we have

$$
\begin{aligned}
\sum_c |\rho_c u \cdot \mathbf{n}|_{H^{1-\frac{1}{2}}(\partial U_c)} &= \sum_c \inf_{\beta|_{\partial U_c} = \rho_c u \cdot \mathbf{n}} |\beta|_{H^1(U_c)} \\
&\leq \sum_c (1 + C_\epsilon) \inf_{\beta|_{\partial \Omega_c} = g(\rho_c u, \mathbf{n})} |\beta|_{H^1(\Omega_c)} \leq (1 + C_\epsilon) \inf_{\phi|_{\partial\Omega} = g(\mathbf{n}, u)} |\phi|_{H^1(\Omega_c)} \quad {}^3
\end{aligned}
$$

so,

$$
|u|_{H^1(T\Omega)} \leq D \left(|u|_{L^2(T\Omega)} + |\nabla \cdot u|_{L^2(\Omega)} + |\nabla^\perp \cdot u|_{L^2(\Omega)} + |g(\mathbf{n}, u)|_{H^{1-\frac{1}{2}}(T\Omega|\partial\Omega)} \right).
$$

[2] We may suppose the boundary of U_c is regular enough. If it is not regular enough we may replace it by a close sub-domain regular enough such that the images of this sub-domain together with the other charts still covering the manifold Ω.

[3] Recall that domains of charts containing pieces of boundaries are diffeomorphic to a "disk" where, half of the disk correspond to the part in Ω the other half to the part outside Ω (in a bigger manifold) and; the line separating these half disks correspond to the piece of boundary. Vector fields tangent to that separating line $g(\tau)$ correspond to vector fields tangent to the piece of boundary $\Phi(g(\tau))$: $\frac{d}{d\tau}(\Phi \circ g)(\tau) = D\Phi_{|g(\tau)} \frac{d}{d\tau} g(\tau)$.

Remark 5.4.1. *In the previous computations the constants C_ϵ are small for small ϵ. For smaller ϵ we may need more charts and consequently more constants D_c but, for a fixed ϵ we have a finite number of these constants so, we may set the biggest one.*

The case $m \geq 2$ differs from the case $m = 1$ on the fact that, derivatives of order bigger than one of the metric coefficients g_{ij} do not vanish at the center of each neighborhood Ω_c but, since the neighborhoods can be set such that those derivatives, up to order m, are bounded by some constant K_m on every neighborhood, we may arrive to

Proposition 5.4.1.

$$\begin{aligned} & H^m(T\Omega) \\ =& \Big\{ u \in L^2(T\Omega) \mid \nabla \cdot u \in H^{m-1}(\Omega) \ \ \& \ \ \nabla^\perp \cdot u \in H^{m-1}(\Omega) \ \ \& \ \ g(\mathbf{n},\, u) \in H^{m-\frac{1}{2}}(T\Omega \mid \partial\Omega) \Big\} \end{aligned}$$

and

$$|u|_{H^m(T\Omega)} \leq D_m \Big(|u|_{L^2(T\Omega)} + |\nabla \cdot u|_{H^{m-1}(\Omega)} + |\nabla^\perp \cdot u|_{H^{m-1}(\Omega)} + |g(\mathbf{n},\, u)|_{H^{m-\frac{1}{2}}(T\Omega|\partial\Omega)} \Big).$$

Moreover in the case Ω is simply-connected and $\partial\Omega \neq \emptyset$, we may omit the term $|u|_{L^2(T\Omega)}$ in the last estimate.

The last statement holds because in the simply-connected case each vector field can be recovered by its divergence, vorticity and normal component. Recovering not possible means that there would exist a nonzero vector field u with vanishing divergence, vorticity and normal component. The vanishing of the divergence implies that $d * u^\sharp = 0$. By the Poincaré's Theorem, since $\partial\Omega$ is contractible to a point, we have that $*u^\sharp$ is exact on Ω, i.e., $*u^\sharp = d\xi$ (and then $-u = \nabla^\perp \xi$) for some function ξ. Thus $g(\nabla\xi,\, \mathbf{t}) = -g(u,\, \mathbf{n}) = 0$, so ξ is constant on the boundary and changing it by a constant we suppose it is zero on the boundary. By $\Delta\xi = \nabla^\perp \cdot u = 0$ we have $\xi = 0$ which implies $u = 0$.

We check the validity of the proposition only for $m = 2$ the case we will need in this study, for other $m \geq 2$ we may proceed similarly. We compute the second covariant

$$\begin{aligned} \nabla^2 u = & \Big(\partial_s \partial_k u^i + (\partial_s u^r)\Gamma^i_{rk} + u^r(\partial_s \Gamma^i_{rk}) \Big) dx^s \otimes dx^k \otimes \partial_i \\ & + \Big(\partial_k u^i + u^r \Gamma^i_{rk} \Big) \Big(-\Gamma^k_{pq} dx^p \otimes dx^q \otimes \partial_i + \Gamma^t_{ij} dx^k \otimes dx^j \otimes \partial_t \Big) \end{aligned}$$

or

$$\nabla^2 u = \Big[\partial_s \partial_k u^i + (\partial_s u^r)\Gamma^i_{rk} + u^r(\partial_s \Gamma^i_{rk}) - (\partial_q u^i + u^r \Gamma^i_{rq})\Gamma^q_{sk} + (\partial_s u^t + u^r \Gamma^t_{rs})\Gamma^i_{tk} \Big] dx^s \otimes dx^k \otimes \partial_i.$$

In each neighborhood Ω_c, the products $g(A,\, B)$ between the simple tensors A and B, where the coefficients of A and B are not in the family $\{\partial_s \partial_k u^i,\, u^r(\partial_s \Gamma^i_{rk})\}$, are bounded by $C_\epsilon(|\partial_i u^j|^2 + |u^i|^2)$; products between tensors with coefficients of the kind $\partial_s \partial_k u^i$ are bounded by $(1 + C_\epsilon)|\partial_s \partial_k u^i|^2$; products between tensors of the kind $u^r(\partial_s \Gamma^i_{rk})$ are bounded by $K|u^r|^2$ for some constant depending on the bound K_2 of the derivatives of the elements g_{ij} up to order 2. Considering also the mixed products we arrive to

$$g(\nabla^2 u,\, \nabla^2 u) \leq C|\partial_s \partial_k u^i|^2 + C_\epsilon |\partial_k u^i|^2 + K|u^i|^2$$

where only K depends on the bound the derivatives of the metric coefficients g_{ij}. In particular, proceeding as in the case $m = 1$, we obtain

$$
\begin{aligned}
&|u|_{H^2(T\Omega)} \\
\leq &\sum_c D_c\Big(|\rho_c u|_{(L^2(U_c))^2} + |\nabla\cdot(\rho_c u)|_{H^1(U_c)} + |\nabla^\perp\cdot(\rho_c u)|_{H^1(U_c)} + |\rho_c u\cdot\mathbf{n}|_{H^{2-\frac{1}{2}}(\partial U_c)}\Big) \\
\leq &\sum_c E_c\Big(|\rho_c u|_{L^2(T\Omega_c)} + |\nabla\cdot(\rho_c u)|_{H^1(\Omega_c)} + |\nabla^\perp\cdot(\rho_c u)|_{H^1(\Omega_c)} + |g(\rho_c u,\,\mathbf{n})|_{H^{2-\frac{1}{2}}(\partial\Omega_c)}\Big)
\end{aligned}
$$

The constants D_c and E_c depend on the bound of the derivatives of order two of the metric coefficients g_{ij}.

5.5 The operators and spaces

We "project" the equation onto the space of divergence-free vector fields tangent to the boundary obtaining

$$u_t = -\nu\tilde{A}u - Bu + F + v.$$

Actually we may project when both sides of the equation are in $L^2(T\Omega)$, this is the case for regular enough external forces and initial condition. As in the Euclidean case we split $\tilde{A}$ as $A - C$ in order to have the desired properties for A.

5.5.1 The space H

We denote by H the space of divergence free vector fields tangent to the boundary Γ of Ω:

$$H := \{v \in L^2(T\Omega) \mid \nabla\cdot v = 0 \text{ on } \Omega,\ g(\mathbf{n},\,v) = 0 \text{ on } \Gamma\}.$$

We have the following theorem which proof may be found in [8, section I.8]:

Theorem 5.5.1. *For a compact Riemannian manifold Ω with boundary Γ:*

1. *Given a function f and $v \in H$, $\int_\Omega i_v\, d\Omega \wedge df = 0$;*
2. *If $\int_\Omega i_v\, d\Omega \wedge w = 0$ for all $v \in H$, then $w = df$ for some function f;*
3. *If $\int_\Omega i_v\, d\Omega \wedge w = 0$ for all $w = df$, then $v \in H$.*

Now by

$$0 = i_v(d\Omega \wedge w) = i_v d\Omega \wedge w + (-1)^n w(v) d\Omega,$$

where n is the dimension of Ω, we conclude that the space orthogonal to H in $L^2(T\Omega)$ is the space

$$H^\perp = \{\nabla f \in L^2(T\Omega) \mid f \text{ a function}\}.$$

The orthogonal projection map from $L^2(T\Omega)$ onto H will be denoted by P^∇.

5.5.2 The operators A and C

We define the compact operator A and the the subspaces V and $D(A)$ analogously as in the Euclidean case. H, V and $D(A)$ are the domains of the operators A^0, $A^{\frac{1}{2}}$ and A. Also C is defined analogously.

5.5.3 The operator B

Similarly as we have done for Euclidean domains, for any Riemannian manifold, we define the trilinear form

$$b(u,\, v,\, w) := \int_\Omega g(\nabla^1_u v,\, w)\, d\Omega;$$

for vector fields u, v, w.

Since

$$\int_\Omega a g(b,\, c)\, d\Omega = \int_\Omega d(g(b,\, c))(a)\, d\Omega = \int_\Omega g(\nabla(g(b,\, c)),\, a)\, d\Omega = 0$$

for any $a \in H$, from the equality

$$g(\nabla^1_u v,\, w) = ug(v,\, w) - g(v,\, \nabla^1_u w)$$

we have

Corollary 5.5.2. *Fixed the first variable $u \in H$, the form b is skew-symmetric in the last two:*

$$b(u,\, v,\, w) = -b(u,\, w,\, v).$$

The desired estimates[4] for the trilinear form also hold in our manifold Ω. Locally, on Ω_c, we replace $g(\nabla^1_u v,\, w)$ by $g_{jp}\rho_c u^i \partial_i \rho_c v^j \rho_c w^p + g_{kp}\rho_c u^i \rho_c v^j \rho_c w^p \Gamma^k_{ij}$; going to the Euclidean image U_c of Ω_c the integrals corresponding to the terms $g_{jp}\rho_c u^i \partial_i \rho_c v^j \rho_c w^p$ are bounded by $(1 + C_\epsilon)E$, where E is one of the desired estimates; similarly those integrals corresponding to $g_{kp}\rho_c u^i \rho_c v^j \rho_c w^p \Gamma^k_{ij}$ are bounded by $C_\epsilon E$ because the operator $\tilde{b}(u,\, v,\, w) := \int_{U_c} u^i v^j w^k dx$ clearly satisfies all the same estimates $b(u,\, v,\, w) = \int_{U_c} u^i \partial_i v^j w^j dx$ does in U_c. Going back to Ω_c we conclude that the desired estimates for the trilinear form are also true in our compact manifold Ω.

As soon as the map $w \mapsto b(u,\, v,\, w)$ is continuous on V we may define $B(u,\, v) \in V'$ by $\langle B(u,\, v),\, w\rangle_{V',V} := b(u,\, v,\, w)$. If $B(u,\, v) \in H$ we have

$$B(u,\, v) \equiv P^\nabla \nabla^1_u v.$$

For simplicity put $Bu := B(u,\, u)$.

5.6 Saturation

We may proceed as in the Euclidean case to derive results concerning existence, uniqueness, continuity of solutions of the Navier-Stokes equation.[5]

We may define analogously the notions of V-saturating set and l-saturating set. At last we arrive to the conclusion that the existence of a V-saturating set is sufficient condition for either either H-approximate controllability or controllability on finite-dimensional observed component.

[4] See ch. 4 or ch. 1.

[5] At this point we should refer to [17] where we may find results concerning Neumann problems in Riemannian manifolds. In particular from [17, Lemme 1] the solution of the Neumann problem $\Delta\phi = \nabla \cdot v$, with $g(\nabla\phi,\, \mathbf{n}) = g(v,\, \mathbf{n})$ on the boundary, is smooth for smooth v, when $\int_\Omega \nabla \cdot v\, d\Omega = -\int_{\partial\Omega} g(v,\, \mathbf{n})\, d\partial\Omega$.

Denote the Poisson bracket between two functions f, h by $\{f, h\}$; consider the closed subspace $\mathcal{S} \subset L^2(T\Omega)$

$$\mathcal{S} := \{\nabla^{\perp}\psi \mid \psi \in H^1(\Omega)\}$$

and its orthogonal

$$\mathcal{S}^{\perp} := \{u \in L^2(T\Omega) \mid \nabla^{\perp} \cdot u = 0 \text{ in } \Omega,\, u \cdot \mathbf{t} = 0 \text{ on } \partial\Omega\},$$

are closed in $L^2(T\Omega)$. Analogously to the Euclidean case:

Definition 5.6.1. *Consider a finite set $h = \{v_i \mid i = 1, \ldots, p\} \subset (\nabla^{\perp} \cdot H_E)$, where H_E is the subset of $H \cap \mathcal{S}$ consisting of steady states of the Euler equation. The set h is said* $l^{\perp}$**-saturating** *if the sequence $(L^{\perp,j})_{j\in\mathbb{N}}$ of finite-dimensional subspaces defined recursively by*

1. $L^{\perp,0} := span(h)$;
2. $L^{\perp,m+1} := L^{\perp,m} + span\{\{\Delta^{-1}v_i,\, v\} + \{\Delta^{-1}v,\, v_i\} \mid i = 1, \ldots, p,\, v \in L^{\perp,m}\}$

satisfies,

$$\overline{\bigcup_{i\in\mathbb{N}} L^{\perp,i}} = \overline{\nabla^{\perp} \cdot (V \cap \mathcal{S})}$$

where the closure is to be taken in $L^2(\Omega)$-norm.

Vector fields u in $H \cap \mathcal{S}$ may be written as $u = -\nabla^{\perp}\psi$ for some function ψ. This function is unique up to an additive constant. If the boundary $\partial\Omega$ of the manifold Ω is nonempty we select that ψ vanishing at the boundary $\partial\Omega$; if the boundary is empty we select that ψ with zero average $\int_{\Omega} \psi \, d\Omega = 0$. The function ψ so selected is called the **stream function** for the solenoidal vector field u.

As usual, denote the Lie bracket between vector fields u, v by $[u, v]$.

Theorem 5.6.1. *For $u \in H$ (such that also $Bu \in H$):*

$$Bu := P^{\nabla}\nabla_u u \equiv -P^{\nabla} \int_{\Omega} (u, [u, \cdot]) \, d\Omega. \quad [6]$$

Proof. Given $w \in H$,

$$\begin{aligned} g(\nabla_u u,\, w) = ug(u,\, w) - g(u,\, \nabla_u w) &= ug(u,\, w) + g(u,\, [w, u]) - g(u,\, \nabla_w u) \\ &= ug(u,\, w) + g(u,\, [w, u]) - \frac{1}{2} wg(u,\, u). \end{aligned}$$

Then $(Bu,\, w)_{L^2(T\Omega)} = \int_{\Omega} g(\nabla_u u,\, w) \, d\Omega = \int_{\Omega} -g(u,\, [u, w]) \, d\Omega$. □

Now, for $u,\, v \in H$ such that the integral $\int_{\Omega} g(v,\, [u, w]) \, d\Omega$ is finite for all $w \in H$, denote by $B_L(u,\, v)$ the element in H defined by

$$(B_L(u,\, v),\, w)_{L^2(T\Omega)} := -\int_{\Omega} g(v,\, [u, w]) \, d\Omega.$$

Denote also $B_L u := B_L(u,\, u)$.

[6] Recall that we identify H with its dual H'.

Theorem 5.6.2. $B_L(u, v) := -P^{\nabla}(i_u dv^{\sharp})^{\flat}$.

The proof may be found in [9].

From theorem 5.6.2 we may derive

Corollary 5.6.3. *If ψ_u, ψ_v are stream functions for $u, v \in H$, then we have*

$$B_L(u, v) = -P^{\nabla}(\Delta\psi_v \nabla\psi_u).$$

Proof. For any $u, v, w \in H$

$$\begin{aligned} i_u dv^{\sharp}(w) &= i_w i_u dv^{\sharp} = -i_w i_u d * d\psi_v = -i_w i_u * *d * d\psi_v = i_w i_u * \Delta\psi_v \\ &= \Delta\psi_v i_w i_u * (1) = \Delta\psi_v i_w * u^{\sharp} \\ &= -\Delta\psi_v i_w * *d\psi_u = \Delta\psi_v d\psi_u(w) = g(\Delta\psi_v \nabla\psi_u, w). \end{aligned}$$

Then

$$(B_L(u, v), w)_{L^2(T\Omega)} = -\int_\Omega i_u dv^{\sharp}(w)\, d\Omega = -\int_\Omega g(\Delta\psi_v \nabla\psi_u, w)\, d\Omega.$$

□

From

$$\begin{aligned} \nabla^{\perp} \cdot B_L(u, v) &= -g(\nabla^{\perp}[\Delta\psi_u], \nabla\psi_v) - \Delta\psi_u(\nabla^{\perp} \cdot \nabla\psi_v) = -g(\nabla^{\perp}[\Delta\psi_u], \nabla\psi_v); \\ \Delta\psi_u &= -\nabla^{\perp} \cdot (\nabla^{\perp}\psi_u) = \nabla^{\perp} \cdot u \end{aligned}$$

and,

$$\{f, h\} := *(df \wedge dh) = i_{\nabla h} * df = g(\nabla^{\perp} f, \nabla h).$$

we have that

$$\nabla^{\perp} \cdot B_L(u, v) = -\{\nabla^{\perp} \cdot u, \Delta^{-1}(\nabla^{\perp} \cdot v)\}$$

and from theorem 5.6.1 we have that $Bw = B_L w$, so by the identity

$$Bu + Bv - B(u, v) - B(v, u) = B(u - v) = B_L(u - v) = B_L u + B_L v - B_L(u, v) - B_L(v, u)$$

we derive that $-B(u, v) - B(v, u) = -B_L(u, v) - B_L(v, u)$ and then

$$\nabla^{\perp} \cdot \Big(- B(u, v) - B(v, u)\Big) = \nabla^{\perp} \cdot \Big(- B_L(u, v) - B_L(v, u)\Big).$$

5.6.1 Simply-connected case (homeomorphic to a disk in the plane)

In the case of a simply-connected manifold we have $H \subset \mathcal{S}$, so

Corollary 5.6.4. *Under Navier boundary conditions, in the simply-connected case, the existence of a $l^{\perp}$-saturating set is a sufficient condition for both H-approximate controllability and controllability on finite-dimensional observed component: from a $l^{\perp}$-saturating set h we may obtain the l-saturating set $g := \left(\nabla^{\perp}\cdot\right)^{-1} h$.*

5.6.2 Multi-connected case (homeomorphic to a disk with a finite number of holes)

In this case we may obtain some results but, we have to take more care, see discussion for the case of Euclidean multi-connected domains in chapter 4.

5.6.3 Empty boundary case

In the case of a two dimensional compact manifold Ω without boundary, we have no problems concerning boundary conditions.

Corollary 5.6.5. *In the boundaryless case, the existence of a $l^\perp$-saturating set is a sufficient condition for both $H \cap \mathcal{S}$-approximate controllability and controllability on finite-dimensional observed component in $H \cap \mathcal{S}$: from a $l^\perp$-saturating set h we may obtain the "l-saturating" set $g := \left(\nabla^\perp \cdot\right)^{-1} h$.*

By "l-saturating" we mean that the saturation is relative to $H \cap \mathcal{S}$.

The vorticity of a harmonic vector field is a harmonic function; since we are in the case of a compact manifold without boundary, that function is necessarily constant because the vanishing of the Laplacean of a function f implies that $0 = (\Delta f,\, f)_{L^2(T\Omega)} = (\delta df,\, f)_{L^2(T\Omega)}$, so

$$\begin{aligned} 0 &= -\int_\Omega f(*d*df)\, d\Omega = -\int_\Omega f(d*df) = -\int_\Omega d(f(*df)) + \int_\Omega df \wedge *df \\ &= 0 + \int_\Omega *(df \wedge *df)\, d\Omega = \int_\Omega -i_{\nabla f} **df\, d\Omega = \int_\Omega g(df,\, df)\, d\Omega; \end{aligned}$$

thus $g(df,\, df) = 0$ which gives $df = 0$ and then necessarily f is constant.

Therefore, in the boundaryless case, the space $H \cap \mathcal{S}$ contains no nonzero harmonic vector field: if $u \in H \cap \mathcal{S}$ is harmonic we have that its vorticity $\nabla^\perp \cdot u = c$ is constant; its stream function ξ solves $\Delta\xi = c$. From $(\Delta\xi,\, d) = (\nabla\xi,\, \nabla d) = 0$ for all constant d and, $(c,\, d) = cd\int_\Omega d\Omega$, we conclude that $c = 0$. In other words we are not able to find a stream function for the solenoidal vector fields with constant nonzero vorticity, i.e., for the solenoidal harmonic vector fields.

Then in the boundaryless case the study for the vorticity equation must be done in the subspace orthogonal to constants. Recall that in [4, 6], working with the vorticity equation in the case of the torus $\mathbb{T}^2$, the study has been done in the space orthogonal to the constants. See also [30, section 2].

Remark 5.6.1. *Functions ψ orthogonal to constants are zero averaged functions: $0 = (\psi,\, c) = c\int_\Omega \psi\, d\Omega$. In the space of zero averaged functions ψ we have the following inequalities useful in the study of the vorticity equation*

$$|\psi|^2_{L^2(\Omega)} \le C_1 |\nabla\psi|^2_{L^2(T\Omega)} \le C_2 |\Delta\psi|^2_{L^2(\Omega)}\, :$$

if the first inequality was not true there would exist a sequence ψ_n with $|\psi_n|^2_{L^2(\Omega)} = 1$ and $1 > n|\nabla\psi_n|^2_{L^2(T\Omega)}$; necessarily $|\nabla\psi_n|^2_{L^2(T\Omega)}$ goes to zero and $|\psi_n|^2_{H^1(\Omega)}$ is bounded. Then there exists a subsequence $\psi_{\sigma(n)}$ of ψ_n that converges in $L^2(T\Omega)$; necessarily the limit is a *constant and since the elements of the sequence have average zero, that constant is necessarily* 0 *which contradicts the fact that* $|\psi_{\sigma(n)}|^2_{L^2(T\Omega)} = 1$. *The second inequality follows from the first and from Young inequality: for some constant* C_2 *we have* $|\nabla\psi|^2_{L^2(T\Omega)} = (\Delta\psi, \psi) \leq \frac{C_2}{2}|\Delta\psi|^2_{L^2(\Omega)} + \frac{1}{2C_1}|\psi|^2_{L^2(\Omega)}$.

5.7 Appendix

We recall some basic nomenclature and tools of the theory of Riemannian geometry. For some details we refer to [31] and [57].

5.7.1 Riemannian metric

Let Ω be a n-dimensional smooth Riemannian manifold with boundary Γ. In each point $x \in \Omega$ we have the tangent space $T_x\Omega$. Given local coordinates $(x^1, x^2, \ldots, x^n)$ induce on Ω the vector fields $\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \ldots, \frac{\partial}{\partial x^n}$ forming a basis on the tangent bundle $T\Omega$. We define dx^i, $i = 1, \ldots, n$ as the adjoint basis to $\frac{\partial}{\partial x^i}$, $i = 1, \ldots, n$ on the dual $T^*(\Omega)$, i.e., to each $x \in \Omega$, $\frac{\partial}{\partial x^i}(x) \in T_x\Omega$, $dx^i(x) \in T^*_x(\Omega)$ and $dx^i(x)\left(\frac{\partial}{\partial x^j}(x)\right) = \delta_{ij}$ – the Kronecker delta, taking the value 1 if $i = j$ and the value 0 otherwise.

We have a metric $g = g(x)$, smooth on x, defining a scalar product on each tangent space $T_x\Omega$; in coordinates $(x^1, x^2, \ldots, x^n)$

$$g = g_{ij}dx^i \otimes dx^j.$$

We use the *Einstein summation convention*: Indexes (occurring twice) are to be summed from 1 to the space dimension n.

By the definition of dx^i we have $g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$ and, since g defines a scalar product, $g_{ij} = g_{ji}$ and $g_{ij}v^iv^j > 0$ for any non-zero vector field $v = v^i\frac{\partial}{\partial x^i}$. We also have $\bar{g} := det[g_{ij}] > 0$. We choose the "orientation" $\left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \ldots, \frac{\partial}{\partial x^n}\right)$ as being positive.

The metric g induces an isomorphism $^\sharp$ (with inverse $^\flat$) between the space of smooth vector fields

$$V(\Omega) = \{U : \Omega \to T_x\Omega \mid x \mapsto U(x),\ U \text{ smooth}\}$$

and its "dual" space of 1-forms

$$\Lambda^1(\Omega) = \{\alpha : \Omega \to T^*_x\Omega \mid x \mapsto \alpha(x),\ \alpha \text{ smooth}\} :$$

given a vector field $V \in V(\Omega)$ and a form $\alpha \in \Lambda^1(\Omega)$

$$V^\sharp(W) := g(V, W), \quad g(\alpha^\flat, W) := \alpha(W); \qquad \forall W \in V(\Omega).$$

In coordinates, for $V = v^i\frac{\partial}{\partial x^i}$ and $\alpha = \alpha_i dx^i$ we obtain

$$V^\sharp = g_{ij}v^idx^j; \quad \alpha^\flat = g^{ij}\alpha_i\frac{\partial}{\partial x^j}$$

where $[g^{ij}]$ is the inverse matrix to $[g_{ij}]$.

The scalar product $g(x)$ induces on $T^*_x\Omega$ the scalar product defined, at each x, by

$$g(x)(\alpha, \beta) := g(x)(\alpha^\flat, \beta^\flat);$$

so on $T_x^*\Omega$ we have $g = g^{ij}\frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}$.

The elements of $\Lambda^1(\Omega)$ are called 1-forms and the elements of

$$\Lambda^k(\Omega) = \{\alpha \mid \alpha(x) : (T_x\Omega)^k \to \mathbb{R} \text{ is multilinear and skew-symmetric}\}$$

are called k-forms, $0 \leq k \leq n$ (for $k = 0$ we have functions on Ω). By skew-symmetry we mean

$$\begin{aligned}&\alpha(V_1, \ldots, V_{i-1}, V_i, V_{i+1}, \ldots, V_{j-1}, V_j, V_{j+1}, \ldots, V_k)\\ = &-\alpha(V_1, \ldots, V_{i-1}, V_j, V_{i+1}, \ldots, V_{j-1}, V_i, V_{j+1}, \ldots, V_k), \quad i \neq j,\end{aligned}$$

i.e., if we change the position of two vector fields we get the minus sign. We suppose the reader is familiar with the classical operations on the space of forms, namely the *wedge product* between forms: $\alpha \wedge \beta$; the *differential* of a form: $d\alpha$ and the *interior product* between a vector field u and a form α: $i_u\alpha$). For some properties see [16, 31, 57].

5.7.2 The Hodge map

The natural volume element on Ω is given by $d\Omega := \sqrt{g}dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n$.

The Hodge map $*$ is a map sending k-forms to $(n-k)$-forms, $0 \leq k \leq n$:

$$\begin{aligned}* : \Lambda^k(\Omega) &\to \Lambda^{n-k}(\Omega)\\ w &\mapsto *w\end{aligned}$$

defined by

$$*w(V_{k+1}, \ldots, V_n)d\Omega = w \wedge V_{k+1}^\sharp \wedge \cdots \wedge V_n^\sharp.$$

Easily we obtain the following properties:

$$\begin{aligned}*(a_1w_1 + a_2w_2) &= a_1 * w_1 + a_2 * w_2, \quad \text{for functions } a_1, a_2 \text{ and, } k\text{-forms } w_1, w_2;\\ i_V * w &= *(w \wedge V^\sharp).\end{aligned}$$

Since any k-form is a combination of elements of the form

$$\beta = a(x)dx^{\sigma(1)} \wedge dx^{\sigma(2)} \wedge \cdots \wedge dx^{\sigma(k)},$$

it is important to know what is $*(dx^{\sigma(1)} \wedge dx^{\sigma(2)} \wedge \cdots \wedge dx^{\sigma(k)})$. From the definition:

$$\begin{aligned}&*(dx^{\sigma(1)} \wedge dx^{\sigma(2)} \wedge \cdots \wedge dx^{\sigma(k)})(V_{k+1}, \ldots, V_n)d\Omega\\ =&dx^{\sigma(1)} \wedge dx^{\sigma(2)} \wedge \cdots \wedge dx^{\sigma(k)} \wedge V_{k+1}^\sharp \wedge \cdots \wedge V_n^\sharp\end{aligned}$$

and, taking the value at $(\frac{\partial}{\partial x^{\sigma(1)}}, \ldots, \frac{\partial}{\partial x^{\sigma(k)}}, \frac{\partial}{\partial x^{\sigma(k+1)}}, \ldots, \frac{\partial}{\partial x^{\sigma(n)}})$, where σ is a permutation of $\{1, \ldots, n\}$, we obtain

$$\begin{aligned}&*(dx^{\sigma(1)} \wedge dx^{\sigma(2)} \wedge \cdots \wedge dx^{\sigma(k)})(V_{k+1}, \ldots, V_n)\sqrt{g}sign(\sigma)\\ =&\det\begin{bmatrix} Id_k & 0\\ V_{k+i}^\sharp(\frac{\partial}{\partial x^{\sigma(p)}}) & V_{k+i}^\sharp(\frac{\partial}{\partial x^{\sigma(k+j)}})\end{bmatrix} \qquad (1 \leq i,j \leq n-k,\ 1 \leq p \leq k)\\ =&\det\left[V_{k+i}^\sharp(\frac{\partial}{\partial x^{\sigma(k+j)}})\right], \qquad 1 \leq i,j \leq n-k\\ =&\frac{\partial}{\partial x^{\sigma(k+1)}}^\sharp \wedge \cdots \wedge \frac{\partial}{\partial x^{\sigma(n)}}^\sharp(V_{k+1}, \ldots, V_n).\end{aligned}$$

Therefore

$$*(dx^{\sigma(1)} \wedge dx^{\sigma(2)} \wedge \cdots \wedge dx^{\sigma(k)}) = \frac{sign(\sigma)}{\sqrt{g}} \frac{\partial}{\partial x^{\sigma(k+1)}}^{\sharp} \wedge \cdots \wedge \frac{\partial}{\partial x^{\sigma(n)}}^{\sharp}.$$

Similarly we compute $*\left(\frac{\partial}{\partial x^{\sigma(k+1)}}^{\sharp} \wedge \cdots \wedge \frac{\partial}{\partial x^{\sigma(n)}}^{\sharp}\right)$: By the definition and taking the value at

$$(dx^{\sigma(k+1)^{\flat}}, \ldots, dx^{\sigma(n)^{\flat}}, dx^{\sigma(1)^{\flat}}, \ldots, dx^{\sigma(k)^{\flat}})$$

we arrive to

$$\begin{aligned}
&*\left(\frac{\partial}{\partial x^{\sigma(k+1)}}^{\sharp} \wedge \cdots \wedge \frac{\partial}{\partial x^{\sigma(n)}}^{\sharp}\right)(V_1, \ldots, V_k)\sqrt{g}g^{-1}(-1)^{k(n-k)}sign(\sigma) \\
&= \det \begin{bmatrix} Id_{n-k} & 0 \\ V_i^{\sharp}(dx^{\sigma(k+j)^{\flat}}) & V_i^{\sharp}(dx^{\sigma(p)^{\flat}}) \end{bmatrix} \qquad (1 \leq i, p \leq k,\ 1 \leq j \leq n-k) \\
&= \det \left[V_i^{\sharp}(dx^{\sigma(p)^{\flat}})\right], \qquad 1 \leq i, p \leq k \\
&= dx^{\sigma(1)} \wedge \cdots \wedge dx^{\sigma(k)}(V_1, \ldots, V_k).
\end{aligned}$$

Therefore

$$*\left(\frac{\partial}{\partial x^{\sigma(k+1)}}^{\sharp} \wedge \cdots \wedge \frac{\partial}{\partial x^{\sigma(n)}}^{\sharp}\right) = (-1)^{k(n-k)}\sqrt{g}sign(\sigma)dx^{\sigma(1)} \wedge \cdots \wedge dx^{\sigma(k)}.$$

Important properties of the Hodge map are

$$\begin{aligned} &**\alpha = (-1)^{k(n-k)}\alpha, \quad \alpha \in \Lambda^k(\Omega); \\ &*(1) = d\Omega; \qquad *d\Omega = 1. \end{aligned}$$

Divergence, curl and Laplace-de Rham operators

The **divergence** operator δ is defined, on k-forms, by

$$\delta := (-1)^{n(k+1)+1} * d*;$$

vanishing on functions and, for $k > 0$, sending k-forms to $(k-1)$-forms.

The **Laplace-de Rham** operator Δ is defined by

$$\Delta := d\delta + \delta d;$$

sending k-forms to k-forms.

The **curl** operator $\delta^{\perp}$ is defined by

$$\delta^{\perp} := *d;$$

vanishing on n-forms and, for $k < n$ sending k-forms to $(n-k-1)$-forms.

For a vector field V we define its divergence $\nabla \cdot V$, Laplacean ΔV and vorticity $\nabla^{\perp} \cdot V$ as follows:

$$\nabla \cdot V := \delta V^{\sharp}; \quad \Delta V := (\Delta V^{\sharp})^{\flat}; \quad \nabla^{\perp} \cdot V := \delta^{\perp} V^{\sharp}$$

Remark 5.7.1. *Another classical definition of the divergence* $div\, V$ *of a vector field* V *is* $L_V d\Omega =: (div\, V)d\Omega$, *where* L_V *stays for the Lie derivative on differential forms. It is known (see [3, section 11.4]) that* $L_V \equiv i_V \circ d + d \circ i_V$ *so,*

$$L_V d\Omega = d \circ i_V d\Omega = d * V^\sharp = (*d * V^\sharp) * (1) = (-1)^{n(1+1)+1} \nabla \cdot V d\Omega = -\nabla \cdot V d\Omega,$$

i.e., the two definitions coincide (up to the sign).

We define the **gradient** ∇f of a function f by $\nabla f := (df)^\flat$. In the case the dimension of Ω is $n = 2$, $*df$ is also a 1-form, we define its rotational part $\nabla^\perp f$ by $\nabla^\perp f := (*df)^\flat$.

Example 5.7.1. *In the case* $n = 2$ *from the definitions above we obtain for a function* f *and a vector field* $V = V^i \frac{\partial}{\partial x^i}$*:*

$$\begin{aligned}
df &= \frac{\partial f}{\partial x^i} dx^i; \qquad \nabla f = g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j}; \\
*df &= \frac{1}{\sqrt{g}} \Big(\frac{\partial f}{\partial x^1} \frac{\partial}{\partial x^2}^\sharp - \frac{\partial f}{\partial x^2} \frac{\partial}{\partial x^1}^\sharp \Big) = \frac{1}{\sqrt{g}} \Big(\frac{\partial f}{\partial x^1} g_{2i} dx^i - \frac{\partial f}{\partial x^2} g_{1i} dx^i \Big) \\
&= \frac{1}{\sqrt{g}} \Big(\frac{\partial f}{\partial x^1} g_{2i} - \frac{\partial f}{\partial x^2} g_{1i} \Big) dx^i; \\
\nabla^\perp f &= \frac{1}{\sqrt{g}} g^{ij} \Big(\frac{\partial f}{\partial x^1} g_{2i} - \frac{\partial f}{\partial x^2} g_{1i} \Big) \frac{\partial}{\partial x^j} = \frac{1}{\sqrt{g}} \Big(\frac{\partial f}{\partial x^1} \frac{\partial}{\partial x^2} - \frac{\partial f}{\partial x^2} \frac{\partial}{\partial x^1} \Big);
\end{aligned}$$

$$\begin{aligned}
*V^\sharp &= g_{ij} V^i * dx^j = \frac{1}{\sqrt{g}} V^i \left(g_{1i} \frac{\partial}{\partial x^2}^\sharp - g_{2i} \frac{\partial}{\partial x^1}^\sharp \right) = \sqrt{g}(-V^2 dx^1 + V^1 dx^2); \\
d * V^\sharp &= \frac{\partial}{\partial x^i} (\sqrt{g} V^i) dx^1 \wedge dx^2; \qquad \nabla \cdot V = -\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left(\sqrt{g} V^i \right) \\
dV^\sharp &= \left(\frac{\partial}{\partial x^1} (g_{i2} V^i) - \frac{\partial}{\partial x^2} (g_{i1} V^i) \right) dx^1 \wedge dx^2; \\
\nabla^\perp \cdot V &= \frac{1}{\sqrt{g}} \left(\frac{\partial}{\partial x^1} (g_{i2} V^i) - \frac{\partial}{\partial x^2} (g_{i1} V^i) \right).
\end{aligned}$$

For a function f *we have*

$$\Delta f = \delta df = \nabla \cdot \nabla f = -\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left(\sqrt{g} g^{ji} \frac{\partial f}{\partial x^j} \right).$$

For the function f *and vector field* V *we may also write (again for n=2):*

$$\begin{aligned}
\Delta f &= -\delta^\perp \delta^\perp f = -\nabla^\perp \cdot \nabla^\perp f; \\
\Delta V &= \left((d\delta + \delta d) V^\sharp \right)^\flat = \left((\nabla(\nabla \cdot V))^\sharp - \left(\nabla^\perp (\nabla^\perp \cdot V) \right)^\sharp \right)^\flat = \nabla(\nabla \cdot V) - \nabla^\perp (\nabla^\perp \cdot V).
\end{aligned}$$

5.7.3 The Stokes theorem

Let Γ be the boundary of Ω. A well known theorem is

Theorem 5.7.1 (Stokes theorem). *Given a $(n-1)$-form w we have*

$$\int_\Omega dw = \int_\Gamma w.$$

We denote by $\mathbf{n}$ the unit vector field normal to Γ. The natural volume element on Γ is given by

$$d\Gamma := i_{\mathbf{n}} d\Omega.$$

By definition $\mathbf{n}$ is orthogonal to each vector $V \in T\Gamma$; on the other side we put $d\mathbf{n} := \mathbf{n}^\sharp$ and have $d\mathbf{n} \wedge d\Gamma = f d\Omega$ for some function f. Since $d\Gamma = *\mathbf{n}^\sharp$, we have

$$f = *(\mathbf{n}^\sharp \wedge *\mathbf{n}^\sharp) = (-1)^{n-1} * (*\mathbf{n}^\sharp \wedge \mathbf{n}^\sharp) = (-1)^{n-1} i_{\mathbf{n}} * *\mathbf{n}^\sharp = 1.$$

Divergence and curl theorems

Theorem 5.7.2. *Given a 1-form w we have*

$$\int_\Omega \delta w \, d\Omega = -\int_\Gamma g(w, \, d\mathbf{n}) \, d\Gamma.$$

Proof. Since w is a 1-form, we have that $(\delta w) d\Omega = (- * d * w) d\Omega = -d * w$. Thus

$$\int_\Omega \delta w d\Omega = -\int_\Omega d * w = -\int_\Gamma *w$$

and from

$$\begin{aligned} *w &= i_{\mathbf{n}}(d\mathbf{n} \wedge *w) = i_{\mathbf{n}} * *(d\mathbf{n} \wedge *w) = (-1)^{n-1} i_{\mathbf{n}} * i_{\mathbf{n}} * *w = i_{\mathbf{n}} * i_{\mathbf{n}} w \\ &= w(\mathbf{n}) i_{\mathbf{n}} d\Omega, \end{aligned}$$

we have $\int_\Omega \delta w \, d\Omega = -\int_\Omega g(w, \, d\mathbf{n}) \, d\Gamma$. □

Theorem 5.7.3. *Given a $(n-1)$-form w we have*

$$\int_\Omega \delta^\perp w d\Omega = \int_\Gamma *(d\mathbf{n} \wedge w) \, d\Gamma.$$

Proof. Since w is a $(n-1)$-form, we have that $(*dw) d\Omega = dw$. From $w = (-1)^{n-1} * *w$ and, proceeding as in the proof of the theorem 5.7.2 we see that, on Γ the form w coincides with $(-1)^{n-1} i_{\mathbf{n}}(*w) \, d\Gamma = *(d\mathbf{n} \wedge w) \, d\Gamma$. □

5.7.4 Levi-Civita connection

A **linear connection** on the tangent bundle $T\Omega$ of a manifold Ω gives us a notion of derivative of vector fields and is defined as a map

$$\begin{aligned} D : V(\Omega) &\to V(\Omega) \otimes \Lambda^1(\Omega) \quad \Big(\equiv \Lambda^{1*}(\Omega) \otimes V^*(\Omega)\Big) \\ X &\mapsto DX \end{aligned}$$

with the properties

$$\begin{aligned} DX(\cdot,\, V+W) &= DX(\cdot,\, V) + DX(\cdot,\, W), \quad V, W \in V(\Omega); \\ DX(\cdot,\, fV) &= fDX(\cdot,\, V), \quad V \in V(\Omega),\ f \text{ a function}; \\ D(X+Y)(\cdot,\, V) &= D(X)(\cdot,\, V) + D(Y)(\cdot,\, V), \quad V \in V(\Omega); \\ D(fX)(\cdot,\, V) &= fD(X)(\cdot,\, V) + V(f)X, \quad V \in V(\Omega),\ f \text{ a function}. \end{aligned}$$

From now we denote the vector field $DX(\cdot,\, V)$ by $D_V X$.

A **torsion-free connection** on $T\Omega$ is a connection satisfying

$$D_X Y - D_Y X = [X, Y]$$

where $[X, Y] = X^j \frac{\partial Y^i}{\partial x^j} \frac{\partial}{\partial x^i} - Y^j \frac{\partial X^i}{\partial x^j} \frac{\partial}{\partial x^i}$ is the Lie bracket of the vector fields X and Y.

On a Riemannian manifold with metric g on $T\Omega$, a **metric connection** on $T\Omega$ is a connection satisfying

$$Xg(V,\, W) = g(D_X V,\, W) + g(V,\, D_X W)$$

It turns out that

Theorem 5.7.4. *On each Riemannian manifold $(\Omega,\, g)$ there is precisely one linear torsion-free and metric connection D on $T\Omega$. It is determined by*

$$\begin{aligned} &g(D_X Y,\, Z) \\ &\quad = \frac{1}{2}\Big(Xg(Y,\, Z) - Zg(X,\, Y) + Yg(Z,\, X) - g(X,\, [Y, Z]) + g(Z,\, [X, Y]) + g(Y,\, [Z, X]) \Big) \end{aligned} \tag{5.4}$$

For the proof see [31, section 3.3].

Definition 5.7.1. *The unique connection of theorem 5.7.4 is called* **Levi-Civita connection**.

Chapter 6

Examples

We know examples of l-saturating for four types of domains: The Torus, the Sphere, the Hemisphere and the Euclidean Rectangle.

6.1 The Torus

This case has been well studied in [4, 5] where in particular it has been proven that the set

$$\left\{\begin{pmatrix}0\\1\end{pmatrix}\sin(x_1),\ \begin{pmatrix}0\\-1\end{pmatrix}\cos(x_1),\ \begin{pmatrix}-1\\1\end{pmatrix}\sin(x_1+x_2),\ \begin{pmatrix}1\\-1\end{pmatrix}\cos(x_1+x_2)\right\}$$

is V-saturating. Here we check that this set of vectors is also l-saturating. Actually in [4] the study is done for the vorticity equation, "translating" the result for the vector equation we obtain the set of vector controls above: to the complete family

$$\left\{\sin(k\cdot x),\ \cos(k\cdot x)\ \middle|\ k\in\mathbb{Z}^2\setminus\{(0,\,0)\}\right\}$$

of admissible vorticities, correspond the family

$$\left\{\frac{1}{k_1^2+k_2^2}\sin(k\cdot x),\ \frac{1}{k_1^2+k_2^2}\cos(k\cdot x)\ \middle|\ k\in\mathbb{Z}^2\setminus\{(0,\,0)\}\right\}$$

of stream functions and; the family

$$\left\{-\frac{1}{k_1^2+k_2^2}\begin{pmatrix}-k_2\\k_1\end{pmatrix}\cos(k\cdot x),\ -\frac{1}{k_1^2+k_2^2}\begin{pmatrix}k_2\\-k_1\end{pmatrix}\sin(k\cdot x)\ \middle|\ k\in\mathbb{Z}^2\setminus\{(0,\,0)\}\right\}$$

of vector fields.

Consider the torus $\mathbb{T}^2=\mathbb{S}^1\times\mathbb{S}^1=]0,\,2\pi]\times]0,\,2\pi]$ and consider zero averaged,[1] periodic divergence free vector fields in it. In the present case we consider the evolution of the equation in the Sobolev subspaces

$$\begin{aligned}H &:= \left\{u\in L^2(T\mathbb{T}^2)\ \middle|\ \int_{\mathbb{T}^2}u(x)\,dx=0,\ u \text{ periodic},\ \nabla\cdot u=0\right\};\\ V &:= \left\{u\in H^1(T\mathbb{T}^2)\ \middle|\ u\in H\right\};\\ D(A) &:= \left\{u\in H^2(T\mathbb{T}^2)\ \middle|\ u\in H\right\};\end{aligned}$$

[1] In the case of the Torus, the zero averaged vector fields are the solenoidal ones that can be recovered by the respective vorticities.

where A coincides with the Laplace-de Rham operator $\Delta = -\partial_1^2 - \partial_2^2$. Recall that when considering the Torus as a subset of $\mathbb{R}^3$ the metric induced in the Torus by the Euclidean one in $\mathbb{R}^3$ is given locally by $\delta_{ij} dx \otimes dx^j$ where $(x^i,\, x^j) \in [0,\, 2\pi[^2$, i.e., it is locally Euclidean.

Since $\cos(-k\cdot x) = \cos(k\cdot x)$ and $\sin(-k\cdot x) = -\sin(k\cdot x)$, we may write

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \sum_{k>(0,\,0)} u^s_{1k}\sin(k\cdot x) + u^c_{1k}\cos(k\cdot x) \\ \sum_{k>(0,\,0)} u^s_{2k}\sin(k\cdot x) + u^c_{2k}\cos(k\cdot x) \end{pmatrix}, \qquad (k \in \mathbb{Z})$$

where the order considered is the lexicographical one: $n < m$ if either $(n_1 < m_1)$ or $(n_1 = m_1 \wedge n_2 < m_2)$. Note that $z = (0,\, 0)$ does not enter the sum because nonzero constants are not zero averaged.

By the divergence free condition, putting $\bar{u}^s_k := \begin{pmatrix} u^s_{1k} \\ u^s_{2k} \end{pmatrix}$ and $\bar{u}^c_k := \begin{pmatrix} u^c_{1k} \\ u^c_{2k} \end{pmatrix}$, we obtain

$$\bar{u}^s_k \cdot k = 0, \quad \bar{u}^c_k \cdot k = 0.$$

Therefore we may write

$$u = \sum_{k>(0,0)} u^s_k \begin{pmatrix} -k_2 \\ k_1 \end{pmatrix} \sin(k\cdot x) + \sum_{k>(0,0)} u^c_k \begin{pmatrix} k_2 \\ -k_1 \end{pmatrix} \cos(k\cdot x).$$

Put $W^s_k := \begin{pmatrix} -k_2 \\ k_1 \end{pmatrix} \sin(k\cdot x)$ and $W^c_k := \begin{pmatrix} k_2 \\ -k_1 \end{pmatrix} \cos(k\cdot x)$. The family

$$\mathcal{W} := \left\{ W^s_k,\, W^c_k \mid k \in \mathbb{Z},\ k > (0,\, 0)\}\right\}$$

is an orthogonal basis for H.

Let us compute $Bu = P^\nabla[(u\cdot\nabla)u]$: we have that

$$\nabla u_1 = \sum_{k>(0,0)} u^s_k(-k_2) \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \cos(k\cdot x) + \sum_{k>(0,0)} u^c_k(-k_2) \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \sin(k\cdot x);$$

$$\nabla u_2 = \sum_{k>(0,0)} u^s_k(k_1) \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \cos(k\cdot x) + \sum_{k>(0,0)} u^c_k(k_1) \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \sin(k\cdot x).$$

Then

$$\begin{aligned} (u\cdot\nabla)u = &\sum_{m,n>(0,0)} u^s_n u^s_m (n \wedge m) \begin{pmatrix} -m_2 \\ m_1 \end{pmatrix} \sin(n\cdot x)\cos(m\cdot x) \\ &+ \sum_{m,n>(0,0)} u^c_n u^c_m (m \wedge n) \begin{pmatrix} -m_2 \\ m_1 \end{pmatrix} \cos(n\cdot x)\sin(m\cdot x); \\ &+ \sum_{m,n>(0,0)} u^s_n u^c_m (n \wedge m) \begin{pmatrix} -m_2 \\ m_1 \end{pmatrix} \sin(n\cdot x)\sin(m\cdot x) \\ &+ \sum_{m,n>(0,0)} u^c_n u^s_m (m \wedge n) \begin{pmatrix} -m_2 \\ m_1 \end{pmatrix} \cos(n\cdot x)\cos(m\cdot x). \end{aligned}$$

From the identities

$$\int_{\mathbb{T}^2} \sin(n\cdot x)\cos(m\cdot x)\cos(k\cdot x)\,dx = 0;$$
$$\int_{\mathbb{T}^2} \sin(n\cdot x)\sin(m\cdot x)\sin(k\cdot x)\,dx = 0;$$

$$\int_{\mathbb{T}^2} \sin(n\cdot x)\cos(m\cdot x)\sin(k\cdot x)\,dx = \begin{cases} 2\pi^2 & \text{if} \quad k = n\pm m \\ -2\pi^2 & \text{if} \quad k = -n\pm m \\ 0 & \text{otherwise} \end{cases};$$

$$\int_{\mathbb{T}^2} \cos(n\cdot x)\cos(m\cdot x)\cos(k\cdot x)\,dx = \begin{cases} 2\pi^2 & \text{if} \quad k = n\pm m \\ 2\pi^2 & \text{if} \quad k = -n\pm m \\ 0 & \text{otherwise} \end{cases};$$

$$\int_{\mathbb{T}^2} \sin(n\cdot x)\sin(m\cdot x)\cos(k\cdot x)\,dx = \begin{cases} 2\pi^2 & \text{if} \quad k = \pm(n-m) \\ -2\pi^2 & \text{if} \quad k = \pm(n+m) \\ 0 & \text{otherwise} \end{cases};$$

defining for $z\in\mathbb{Z}^2\setminus\{(0,0)\}$, $[z] := \begin{cases} z & \text{if } z > (0,0) \\ -z & \text{if } z < (0,0) \end{cases}$ and; taking the scalar product with each element of $\mathcal{W}$ we find, for $P^\nabla[(u\cdot\nabla)u]$ the following expression

$$\begin{aligned}
&P^\nabla[(u\cdot\nabla)u] \\
=& \sum_{m,n>(0,0)} \frac{1}{2}u_n^s u_m^s (n\wedge m)\Big[(m\cdot(m+n))\frac{1}{|m+n|^2}W^s_{m+n} \\
&\qquad\qquad + (m\cdot(n-m))\frac{1}{|n-m|^2}W^s_{[n-m]}\Big] \\
&+ \sum_{m,n>(0,0)} \frac{1}{2}u_n^c u_m^c (m\wedge n)\Big[(m\cdot(m+n))\frac{1}{|m+n|^2}W^s_{m+n} \\
&\qquad\qquad + (m\cdot(m-n))\frac{1}{|m-n|^2}W^s_{[m-n]}\Big] \\
&+ \sum_{m,n>(0,0)} \frac{1}{2}u_n^s u_m^c (n\wedge m)\Big[(-m\cdot(m+n))\frac{-1}{|m+n|^2}W^c_{m+n} \\
&\qquad\qquad + (-m\cdot[m-n])\frac{1}{|m-n|^2}W^c_{[m-n]}\Big] \\
&+ \sum_{m,n>(0,0)} \frac{1}{2}u_n^c u_m^s (m\wedge n)\Big[(-m\cdot(m+n))\frac{1}{|m+n|^2}W^c_{m+n} \\
&\qquad\qquad + (-m\cdot[n-m])\frac{1}{|m-n|^2}W^c_{[n-m]}\Big].
\end{aligned}$$

Thus

$$\begin{aligned}
P^{\nabla}[(u\cdot\nabla)u] = & \sum_{n>m>(0,0)} \frac{1}{2} u_n^s u_m^s (n\wedge m)(|m|^2-|n|^2)\frac{1}{|m+n|^2}W_{m+n}^s \\
& + \sum_{n>m>(0,0)} \frac{1}{2} u_n^c u_m^c (m\wedge n)(|m|^2-|n|^2)\frac{1}{|m+n|^2}W_{m+n}^s \\
& + \sum_{n>m>(0,0)} \frac{1}{2} u_n^s u_m^s (n\wedge m)(|n|^2-|m|^2)\frac{1}{|n-m|^2}W_{n-m}^s \\
& + \sum_{n>m>(0,0)} \frac{1}{2} u_n^c u_m^c (m\wedge n)(|m|^2-|n|^2)\frac{1}{|m-n|^2}W_{n-m}^s \\
& + \sum_{m,n>(0,0)} \frac{1}{2} u_n^s u_m^c (n\wedge m)(m(n+m))\frac{1}{|m+n|^2}W_{m+n}^c \\
& + \sum_{m,n>(0,0)} \frac{1}{2} u_n^c u_m^s (m\wedge n)(-m(n+m))\frac{1}{|m+n|^2}W_{m+n}^c \\
& + \sum_{m,n>(0,0)} \frac{1}{2} u_n^s u_m^c (n\wedge m)(-m\cdot[m-n])\frac{1}{|m-n|^2}W_{[m-n]}^c \\
& + \sum_{m,n>(0,0)} \frac{1}{2} u_n^c u_m^s (m\wedge n)(-m\cdot[n-m])\frac{1}{|m-n|^2}W_{[n-m]}^c.
\end{aligned}$$

Changing the roles of m and n in the sixth and eighth sums and summing up:

$$\begin{aligned}
P^{\nabla}[(u\cdot\nabla)u] = & \sum_{n>m>(0,0)} \frac{1}{2}(u_n^c u_m^c - u_n^s u_m^s)(n\wedge m)(|n|^2-|m|^2)\frac{1}{|m+n|^2}W_{m+n}^s \\
& + \sum_{n>m>(0,0)} \frac{1}{2}(u_n^c u_m^c + u_n^s u_m^s)(n\wedge m)(|n|^2-|m|^2)\frac{1}{|m-n|^2}W_{n-m}^s \\
& + \sum_{m,n>(0,0)} \frac{1}{2} u_n^s u_m^c (n\wedge m)(m-n)(m+n)\frac{1}{|m+n|^2}W_{m+n}^c \\
& + \sum_{m,n>(0,0)} \frac{1}{2} u_n^s u_m^c (n\wedge m)(-(m+n)\cdot[m-n])\frac{1}{|m-n|^2}W_{[m-n]}^c;
\end{aligned}$$

i.e.,

$$\begin{aligned}
P^{\nabla}[(u\cdot\nabla)u] = & \sum_{n>m>(0,0)} \frac{1}{2}(u_n^c u_m^c - u_n^s u_m^s)(n\wedge m)(|n|^2-|m|^2)\frac{1}{|m+n|^2}W_{m+n}^s \\
& + \sum_{n>m>(0,0)} \frac{1}{2}(u_n^c u_m^c + u_n^s u_m^s)(n\wedge m)(|n|^2-|m|^2)\frac{1}{|m-n|^2}W_{n-m}^s \\
& + \sum_{n>m>(0,0)} -\frac{1}{2}(u_n^c u_m^s + u_n^s u_m^c)(n\wedge m)(|n|^2-|m|^2)\frac{1}{|m+n|^2}W_{m+n}^c \\
& + \sum_{n>m>(0,0)} \frac{1}{2}(u_n^c u_m^s - u_n^s u_m^c)(n\wedge m)(|n|^2-|m|^2)\frac{1}{|m-n|^2}W_{n-m}^c.
\end{aligned}$$

In particular for two given $n, m > (0,\,0)$ such that $n > m$, $|n|^2 \neq |m|^2$ and $n \wedge m \neq 0$ we obtain

$$B(W^s_m \pm W^s_n) = \mp\frac{1}{2}(n\wedge m)(|n|^2-|m|^2)\frac{1}{|m+n|^2}W^s_{m+n} \pm \frac{1}{2}(n\wedge m)(|n|^2-|m|^2)\frac{1}{|m-n|^2}W^s_{n-m};$$

$$B(W^c_m + W^c_n) = \frac{1}{2}(n\wedge m)(|n|^2-|m|^2)\frac{1}{|m+n|^2}W^s_{m+n} + \frac{1}{2}(n\wedge m)(|n|^2-|m|^2)\frac{1}{|m-n|^2}W^s_{n-m};$$

and

$$B(W^c_m \pm W^s_n) = \mp\frac{1}{2}(n\wedge m)(|n|^2-|m|^2)\frac{1}{|m+n|^2}W^c_{m+n} \mp \frac{1}{2}(n\wedge m)(|n|^2-|m|^2)\frac{1}{|m-n|^2}W^c_{n-m};$$

$$B(W^s_m + W^c_n) = -\frac{1}{2}(n\wedge m)(|n|^2-|m|^2)\frac{1}{|m+n|^2}W^c_{m+n} + \frac{1}{2}(n\wedge m)(|n|^2-|m|^2)\frac{1}{|m-n|^2}W^c_{n-m}.$$

so we obtain vectors spanning $\mathit{span}\{W^s_{n+m},\, W^c_{n+m},\, W^s_{n-m},\, W^c_{n-m}\}$ in the image

$$B(\mathit{span}\{W^s_n,\, W^c_n,\, W^s_m,\, W^c_m\}).$$

Now we are ready to prove that the set

$$g^0 := \left\{\begin{pmatrix}0\\1\end{pmatrix}\sin(x_1),\, \begin{pmatrix}0\\-1\end{pmatrix}\cos(x_1),\, \begin{pmatrix}-1\\1\end{pmatrix}\sin(x_1+x_2),\, \begin{pmatrix}1\\-1\end{pmatrix}\cos(x_1+x_2)\right\}.$$

is l-saturating. Let L^k be the sequence of subspaces given in the definition of l-saturating set. Define the set $\mathcal{V}^j := \left\{k > (0,\,0) \mid \{W^s_k,\, W^c_k\} \subseteq L^j\right\}$.

For $j > 0$, put $g^j := \{W^s_k, W^c_k \mid k > (0,\,0),\, k_1 - j \leq k_2 \leq j+1\}$. We prove that $\mathit{span}(g^j) \subseteq L^{2j+1}$, where $(L^j)_{j\in\mathbb{N}}$ is the sequence given in the definition of l-saturating set. By definition $L^0 := \mathit{span}(g^0)$; on the other hand we have

$$\begin{aligned}
\{(2,\,1),\,(0,\,1)\} &= \{(1,\,1)+(1,\,0),\,(1,\,1)-(1,\,0)\} \subseteq \mathcal{V}^1;\\
\{(1,\,2),\,(3,\,2)\} &= \{(1,\,1)+(0,\,1),\,(2,\,1)+(1,\,1)\} \subseteq \mathcal{V}^2;\\
\{(2,\,2),\,(0,\,2)\} &= \{(1,\,2)+(1,\,0),\,(1,\,2)-(1,\,0)\} \subseteq \mathcal{V}^3;
\end{aligned}$$

so $\mathit{span}\{g^1\} \subseteq L^3$.

Now suppose that for $j \geq 1$ we have $\mathit{span}\{g^j\} \subseteq L^{2j+1}$, i.e., $\{k > (0,\,0) \mid k_1 - j \leq k_2 \leq j+1\} \subseteq \mathcal{V}^{2j+1}$. Then

$$\begin{aligned}
&\Big\{(1,\,j+2),\,(1,\,-j)\Big\} \cup \Big\{(p+1,\,j+2) \mid p \in \{1,\,\dots,\,2j+1\}\setminus\{j+1\}\Big\}\\
&\cup\Big\{(p+1,\,p-j) \mid p \in \{2,\,\dots,\,2j+1\}\setminus\{j\}\Big\} \cup \Big\{(2,\,1-j)\Big\}\\
\equiv\ &\\
&\Big\{(1,\,1)\pm(0,\,j+1)\Big\} \cup \Big\{(p,\,j+1)+(1,\,1) \mid p \in \{1,\,\dots,\,2j+1\}\setminus\{j+1\}\Big\}\\
&\cup\Big\{(p,\,p-j)+(1,\,0) \mid p \in \{2,\,\dots,\,2j+1\}\setminus\{j\}\Big\} \cup \Big\{(1,\,0)+(1,\,1-j)\Big\}\\
\subseteq\ & \mathcal{V}^{2j+2},
\end{aligned}$$

and

$$\begin{aligned} &\Big\{(0,\, j+2),\, (2j+3,\, j+2),\, (j+2,\, j+2),\, (j+1,\, 0)\Big\} \\ \equiv \\ &\Big\{(1,\, j+2)-(1,\, 0),\, (2j+2,\, j+2)+(1,\, 0)\Big\} \\ &\cup\Big\{(j+1,\, j+2)+(1,\, 0),\, (j+2,\, 1)-(1,\, 1)\Big\} \\ \subseteq\ & \mathcal{V}^{2j+3}. \end{aligned}$$

So $\{k > (0,\, 0) \mid k_1 - j - 1 \le k_2 \le j+2\} \subseteq \mathcal{V}^{2j+3}$, i.e., $span\{g^{j+1}\} \subseteq L^{2(j+1)+1}$.

6.2 The Sphere

Examples of $l^{\perp}$-saturating sets in this case was found in [6]. At the end of this section we derive explicitly a l-saturating set of vector fields from one of those $l^{\perp}$-saturating sets.

In the present case we may consider the evolution of the equation in the Sobolev subspaces

$$\begin{aligned} H &:= \big\{u \in L^2(T\mathbb{S}^2) \mid \nabla\cdot u = 0\big\}\,; \\ V &:= \big\{u \in H^1(T\mathbb{S}^2) \mid u \in H\big\}\,; \\ D(A) &:= \big\{u \in H^2(T\mathbb{T}^2) \mid u \in V\big\}\,; \end{aligned}$$

where $A = \Delta$.

But, here we are going to work with the vorticity equation; in this case the study must be done in the space of functions orthogonal to constants.

We treat functions on the Sphere $\mathbb{S}^2 := \{x \in \mathbb{R}^3 \mid |x| = 1\}$, where in coordinates $x = (x_1,\, x_2,\, x_3)$ and $|x| = ({x_1}^2 + {x_2}^2 + {x_3}^2)^{\frac{1}{2}}$ is the Euclidean norm in $\mathbb{R}^3$, as the restrictions to $\mathbb{S}^2$ of homogeneous functions on $\mathbb{R}^3$. The degree of homogeneity is not fixed à priori and is in our disposal. In the Sphere we consider the metric induced by the Euclidean metric in $\mathbb{R}^3$.

Lemma 6.2.1. *Let a, b be smooth functions on $\mathbb{R}^3$. The Poisson bracket of their restrictions to the Sphere $\mathbb{S}^2$ can be computed as follows:*

$$\{a_{|\mathbb{S}^2}, b_{|\mathbb{S}^2}\}(x) = \langle x, \nabla_x a, \nabla_x b\rangle, \tag{6.1}$$

where $\langle l_1, l_2, l_3\rangle$ is the determinant of the 3×3-matrix whose columns are l_1, l_2, l_3 (the "mixed product").

Proof. In the coordinates $(u,\, v) \mapsto (x_1,\, x_2,\, x_3) = \Big(u,\, v,\, (1-u^2-v^2)^{\frac{1}{2}}\Big)$ the area form is given by $d\mathbb{S}^2 = \sqrt{g}du\wedge dv = \frac{1}{x_3}du\wedge dv$ (see for example [16], section I.4.12). On the other side

$$\begin{aligned} da_{|\mathbb{S}^2}\wedge db_{|\mathbb{S}^2} &= \left(\frac{\partial a_{|\mathbb{S}^2}}{\partial u}du + \frac{\partial a_{|\mathbb{S}^2}}{\partial v}dv\right)\wedge\left(\frac{\partial b_{|\mathbb{S}^2}}{\partial u}du + \frac{\partial b_{|\mathbb{S}^2}}{\partial v}dv\right) \\ &= \left(\frac{\partial a_{|\mathbb{S}^2}}{\partial u}\frac{\partial b_{|\mathbb{S}^2}}{\partial v} - \frac{\partial a_{|\mathbb{S}^2}}{\partial v}\frac{\partial b_{|\mathbb{S}^2}}{\partial u}\right)du\wedge dv \end{aligned}$$

Now, from the relations

$$\begin{aligned}\frac{\partial f_{|\mathbb{S}^2}}{\partial u} &= \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial u} + \frac{\partial f}{\partial x_3}\frac{\partial x_3}{\partial u} = \frac{\partial f}{\partial x_1} - \frac{x_1}{x_3}\frac{\partial f}{\partial x_3}\\ \frac{\partial f_{|\mathbb{S}^2}}{\partial v} &= \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial v} + \frac{\partial f}{\partial x_3}\frac{\partial x_3}{\partial v} = \frac{\partial f}{\partial x_2} - \frac{x_2}{x_3}\frac{\partial f}{\partial x_3}\end{aligned}$$

for any function f defined in $\mathbb{R}^3$, we obtain

$$da_{|\mathbb{S}^2} \wedge db_{|\mathbb{S}^2} = \frac{1}{x_3} x \cdot (\nabla_x a \wedge \nabla_x b) du \wedge dv = \frac{1}{x_3}\langle x, \nabla_x a, \nabla_x b\rangle du \wedge dv.$$

and then

$$\{a_{|\mathbb{S}^2}, b_{|\mathbb{S}^2}\}(x) = *(da_{|\mathbb{S}^2} \wedge db_{|\mathbb{S}^2}) = \langle x, \nabla_x a, \nabla_x b\rangle.$$

The chart $(u,\, v) \mapsto \left(u,\, v,\, (1-u^2-v^2)^{\frac{1}{2}}\right)$ covers only the part of the Sphere with $x_3 > 0$ but, we can choose some more analogous charts in order to form an atlas covering all the Sphere. The computation of the Poisson bracket is analogous and lead to the same expression in (6.1). For example in the symmetric chart $(u,\, v) \mapsto \left(u,\, v,\, -(1-u^2-v^2)^{\frac{1}{2}}\right)$ covering $x_3 < 0$ we have similarly

$$da_{|\mathbb{S}^2} \wedge db_{|\mathbb{S}^2} = \frac{1}{x_3} x \cdot (\nabla_x a \wedge \nabla_x b) du \wedge dv = \langle x, \nabla_x a, \nabla_x b\rangle \left(-\frac{1}{x_3} dv \wedge du\right).$$

The area form in this chart is $-\frac{1}{x_3} dv \wedge du$ so, again $\{a_{|\mathbb{S}^2}, b_{|\mathbb{S}^2}\}(x) = \langle x, \nabla_x a, \nabla_x b\rangle$. [2] □

Spherical harmonics, i.e., eigenfunctions of the Laplacean in $\mathbb{S}^2$, are exactly restrictions to $\mathbb{S}^2$ of homogeneous harmonic polynomials on $\mathbb{R}^3$. Let $\rho(x) = ({x_1}^2 + {x_2}^2 + {x_3}^2)^{-1/2}$, the fundamental solution of the Laplace equation in $\mathbb{R}^3$. The Maxwell's theorem (see, for instance Lecture 11 of [10]) states that any spherical harmonic a is an iterated directional derivative of ρ:

$$a = (l_1 \circ \cdots \circ l_n \rho)_{|\mathbb{S}^2},$$

where $l_1, \ldots, l_n \in \mathbb{R}^3$ and the set $\{l_1, \ldots, l_n\}$ is uniquely determined by a.

Linear functions are, of course, harmonic. We denote by $\vec{l}$ the Hamiltonian field on the Sphere associated to the function $x \mapsto \langle l, x\rangle := l \cdot x$, $x \in \mathbb{S}^2$; then $\vec{l}a = \langle x, l, \nabla_x a\rangle$ is the Poisson bracket of the functions $\langle l, x\rangle$ and a restricted to the sphere. Obviously, $\vec{l}$ generates rotation of the Sphere around the axis l. Indeed $\vec{l}a$ is the restriction to the Sphere of the function $\langle x, l, \nabla_x a\rangle = (\nabla_x a) \cdot (x \wedge l)$ defined for $x \in \mathbb{R}^3$ and so, $\vec{l}$ is the restriction to the Sphere of the vector field $R_l(x) = x \wedge l$ in $\mathbb{R}^3$; $R_l(x)$ generates rotation around l with angular velocity $w(x) = |l|$; the tangential velocity has speed $|x \wedge l| = |l||x||\sin\alpha|$ and the circumference around l generated by x has length $2\pi|x||\sin\alpha|$; α is the angle between x and l.

The group of rotations acts (by the change of variables) on the space of harmonic polynomials of fixed degree n. It is well-known that this action is irreducible for any n (see Arnold's book [10] for the elementary proof); in other words, given a nonzero degree n homogeneous harmonic polynomial a, the space

$$span\{\vec{l}_1 \circ \cdots \circ \vec{l}_k a : k \geq 0\}$$

[2] The normal to $\mathbb{S}^2$ is given by $\mathbf{n} \equiv x$. Note that $(\mathbf{n}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1})$ is positively oriented in the ambient space $\mathbb{R}^3$. So the "orientation" of the chart $(u,\, v) \mapsto \left(u,\, v,\, -(1-u^2-v^2)^{\frac{1}{2}}\right)$ is $(\frac{\partial}{\partial v}, \frac{\partial}{\partial u})$.

equals the space of all degree n homogeneous harmonic polynomials.

In general, Poisson bracket $\{a_{1|\mathbb{S}^2}, a_{2|\mathbb{S}^2}\}$ of two polynomials is a polynomial of degree $\deg a_1 + \deg a_2 - 1$, but it is not necessary harmonic even if a_1, a_2 are harmonic.

Given quadratic harmonic polynomial q, for the desired saturation property it is sufficient to show that for any $n \geq 2$ there exists a degree n harmonic polynomial p_n such that $\{q, p_n\}$ is a nonzero harmonic function. We start by the following:

Lemma 6.2.2. *For $x \in \mathbb{R}^3$ we have*

$$\langle x, l_1 \circ l_2 \nabla_x \rho, \nabla_x a\rangle = 3\rho^5 \langle x, l_1\rangle R_{l_2} a + 3\rho^5 \langle x, l_2\rangle R_{l_1} a + \rho^3 [R_{l_1}, l_2] a + \rho^3 [R_{l_2}, l_1] a,$$

for any smooth function a and $l_1, l_2 \in \mathbb{R}^3$.

Proof. We recall that the gradient commutes with directional derivatives: given a vector function $f : \mathbb{R}^3 \to \mathbb{R}^k$ sending $x = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^\top$ to $\begin{bmatrix} f_1 & f_2 & \cdots & f_k \end{bmatrix}^\top$ its gradient $\nabla_x f$ is defined by

$$\nabla_x f := (D_x f)^\top = \begin{bmatrix} \nabla_x f_1 & \nabla_x f_2 & \cdots & \nabla_x f_k \end{bmatrix}.$$

Here $(D_x f)$ stays for the derivative of f and $A^\top$ for the matrix transposed of A, i.e., $\nabla_x f$ is a matrix whose columns are the gradients $\nabla_x f_i = \begin{bmatrix} \partial_1 f_i & \partial_2 f_i & \partial_3 f_i \end{bmatrix}^\top$ of the scalar functions f_i.

For $l \in \mathbb{R}^3$, the directional derivative lf is defined by the product $lf := D_x f l$ of the matrices $D_x f$ and l.

Therefore by a simple computation for any smooth real function $f : \mathbb{R}^3 \to \mathbb{R}$ we obtain

$$\begin{aligned} l\nabla_x f = D_x(\nabla_x f) l &= \begin{bmatrix} \partial_{1,1} f & \partial_{2,1} f & \partial_{3,1} f \\ \partial_{1,2} f & \partial_{2,2} f & \partial_{3,2} f \\ \partial_{1,3} f & \partial_{2,3} f & \partial_{3,3} f \end{bmatrix} l = \begin{bmatrix} \partial_1 D_x f \\ \partial_2 D_x f \\ \partial_3 D_x f \end{bmatrix} l \\ &= \begin{bmatrix} (\partial_1 D_x f) l \\ (\partial_2 D_x f) l \\ (\partial_3 D_x f) l \end{bmatrix} = \begin{bmatrix} \partial_1 (D_x f l) \\ \partial_2 (D_x f l) \\ \partial_3 (D_x f l) \end{bmatrix} = \nabla_x (lf), \end{aligned}$$

where $\partial_{i,j}$ stays for $\partial_i \circ \partial_j$. Thus we have

$$\langle x, \nabla_x l\rho, \nabla_x a\rangle = \langle x, l\nabla_x \rho, \nabla_x a\rangle.$$

Using the identity $\nabla_x \rho = -\rho^3 x$ and $\partial_i(\rho^3 x_j) = 3\rho^2(-\rho^3) x_i x_j + \rho^3 \partial_i x_j$, for $l\nabla_x \rho$ we obtain

$$l\nabla_x \rho = -l(\rho^3 x) = -\left(3\rho^2(-\rho^3) \begin{bmatrix} x_1 x^\top \\ x_2 x^\top \\ x_3 x^\top \end{bmatrix} + \rho^3 Id_3 \right) l = 3\rho^5 < l,\, x > x - \rho^3 l.$$

Then

$$\langle x, \nabla_x l\rho, \nabla_x a\rangle = \langle x, 3\rho^5 < l,\, x > x - \rho^3 l, \nabla_x a\rangle = -\rho^3 \langle x, l, \nabla_x a\rangle = -\rho^3 R_l a.$$

In the following computation we use the Leibnitz rule for the differentiation of multi-linear expressions.

$$\begin{aligned}\langle x, l_1 \circ l_2 \nabla_x \rho, \nabla_x a\rangle &= l_1\langle x, l_2\nabla_x\rho, \nabla_x a\rangle - \langle l_1, l_2\nabla_x\rho, \nabla_x a\rangle - \langle x, l_2\nabla_x\rho, l_1\nabla_x a\rangle \\ &= l_1(-\rho^3 R_{l_2}a) - \Big(l_2\langle l_1, \nabla_x\rho, \nabla_x a\rangle - \langle l_1, \nabla_x\rho, l_2\nabla_x a\rangle\Big) + \rho^3 R_{l_2}(l_1 a) \\ &= l_1(-\rho^3 R_{l_2}a) + \rho^3 R_{l_2}(l_1 a) \\ &\quad - l_2\Big(l_1\langle x, \nabla_x\rho, \nabla_x a\rangle - \langle x, l_1\nabla_x\rho, \nabla_x a\rangle - \langle x, \nabla_x\rho, l_1\nabla_x a\rangle\Big) \\ &\quad + \Big(l_1\langle x, \nabla_x\rho, l_2\nabla_x a\rangle - \langle x, l_1\nabla_x\rho, l_2\nabla_x a\rangle - \langle x, \nabla_x\rho, l_1\circ l_2\nabla_x a\rangle\Big);\end{aligned}$$

since x is collinear with $\nabla_x\rho$ this simplifies as follows

$$\begin{aligned}\langle x, l_1 \circ l_2\nabla_x\rho, \nabla_x a\rangle &= l_1(-\rho^3 R_{l_2}a) - l_2\rho^3 R_{l_1}a + \rho^3 R_{l_1}(l_2 a) + \rho^3 R_{l_2}(l_1 a) \\ &= 3\rho^2\rho^3\left(x^\top l_1\right) R_{l_2}a + 3\rho^2\rho^3\left(x^\top l_2\right) R_{l_1}a \\ &\quad - \rho^3 l_1(R_{l_2}a) - \rho^3 l_2(R_{l_1}a) + \rho^3 R_{l_1}(l_2 a) + \rho^3 R_{l_2}(l_1 a) \\ &= 3\rho^5\langle x, l_1\rangle R_{l_2}a + 3\rho^5\langle x, l_2\rangle R_{l_1}a + \rho^3[R_{l_1}, l_2]a + \rho^3[R_{l_2}, l_1]a.\end{aligned}$$

This ends the proof of lemma 6.2.2. □

Now it is known that for $r = \frac{1}{\rho}$, $r^5 l_1 \circ l_2\rho$ is a harmonic and homogeneous polynomial of degree 2 that coincides with $l_1 \circ l_2\rho$ on $\mathbb{S}^2$ (note that $l_1 \circ l_2\rho$ is harmonic and homogeneous of degree -3). If a is a harmonic homogeneous polynomial of degree $n \geq 1$ we expect $\langle x, l_1 \circ l_2\nabla_x\rho, \nabla_x a\rangle$ to be homogeneous of degree $1 + (-4) + n - 1$ but, not necessarily harmonic; so $r^5\langle x, l_1 \circ l_2\nabla_x\rho, \nabla_x a\rangle$ is expected to be homogeneous of degree $n+1$ but again, not necessarily harmonic. First of all, from lemma 6.2.2, immediately we see that for $l_1 = l_2 = l$

$$\{l \circ l\rho, a\} = \Big(6\rho^5\langle l, x\rangle R_l a\Big)_{|\mathbb{S}^2} = 6\langle l, x\rangle\vec{l}a. \tag{6.2}$$

because, in $\mathbb{R}^3$, translations along l commutes with rotations around l.

Now we find two harmonic homogeneous polynomial q and p of degrees 2 and $n \geq 1$ such that $\{q, p\}$ is a harmonic homogeneous polynomial of degree $n + 1$. Let $l := \begin{bmatrix}0\\0\\1\end{bmatrix}$, $q := r^5 l \circ l\rho = 3(x_3)^2 - r^2$ and $p := Re(x_1 + ix_2)^n$ is the real part of the complex number $(x_1 + ix_2)^n$ that is known to be homogeneous of degree n and harmonic. Note that the polynomial $p = p(x_1, x_2)$ defined on $\mathbb{R}^3$ does not depend on the variable x_3. Since q coincides with $l \circ l\rho$ on $\mathbb{S}^2$ we have that $\{q, p\} = \{l \circ l\rho, p\}$, so by (6.2)

$$\{q, p\} = 6x_3\vec{l}p = 6x_3\langle x, l, \nabla_x p\rangle = 6x_3(x_2\partial_1 p - x_1\partial_2 p). \tag{6.3}$$

From the harmonicity of p follows the harmonicity of $R_l p$ for all $l = (l^1, l^2, l^3)$ because, all $\partial_i p$ are harmonic for all $i = 1, 2, 3$ and $x \wedge l$ is also harmonic; so $\Delta R_l p$ is equal to

$$\begin{aligned}&2\sum_{j=1}^{3}\Big(\partial_j\partial_1 p\partial_j(x_2 l^3 - x_3 l^2) + \partial_j\partial_2 p\partial_j(x_3 l^1 - x_1 l^3) + \partial_j\partial_3 p\partial_j(x_1 l^2 - x_2 l^1)\Big) \\ =&2\left(\partial_2\partial_1 p[l^3 - l^3] + \partial_3\partial_1 p[-l^2 + l^2] + \partial_2\partial_3 p[l^1 - l^1]\right) = 0\end{aligned}$$

From the harmonicity of $6x_3$ and $R_l p$ follows that $\Delta\{q,\, p\} = 12\partial_3 R_l p$ and since $R_l p = x_2\partial_1 p - x_1\partial_2 p$ does not depend on x_3 we have that $\{q, p\}$ is harmonic. From the fact that, by (6.3), $\{q, p\}$ is a sum of monomials of degree $1 + 1 + n - 1 = n + 1$ we have that it is either homogeneous of degree $n + 1$ or zero. To see that $\{q,\, p\}$ is nonzero we note that

$$0 = x_2\partial_1 p - x_1\partial_2 p = \begin{bmatrix}\partial_1 p & \partial_2 p\end{bmatrix}\begin{bmatrix} x_2 \\ -x_1\end{bmatrix}$$

means that $p(x_1,\, x_2)$ is constant on each Sphere $\mathbb{S}^1_s = \{x \in \mathbb{R}^2 \mid |x| = s\}$, $s > 0$. But then p is of degree 0 or null, which contradicts the fact that p is of degree $n \geq 1$.

Theorem 6.2.3. *Any set $h = \{l_1,\, l_2,\, l_3,\, l \circ l\rho,\, c\}$ formed by: three linearly independent spherical harmonics l_1, l_2, l_3; the quadratic spherical harmonic $l \circ l\rho$, where $l = (0,\, 0,\, 1)$ and; a cubic spherical harmonic c is $l^\perp$-saturating.*

For the proof we need the following lemma that is an immediate consequence of the irreducibility of the $2k + 1$ dimensional space $\mathcal{P}_k$, of degree-k polynomials, under the group of rotations of the Sphere (see [10]).

Lemma 6.2.4. *Given a linearly independent set $Q := \{q_1,\, q_2,\, \cdots,\, q_s\} \subseteq \mathcal{P}_k$. If $1 \leq s \leq 2k$, there is $l \in \mathbb{S}^2$ such that the set $Q \cup \vec{l}Q$ has at least $s + 1$ linearly independent degree-k polynomials.*

Proof of theorem 6.2.3. Let $(L^{\perp,j})_{j\in\mathbb{N}}$ be the sequence given in the definition of $l^\perp$-saturating set. Obviously

- all the linear harmonics are in $L^{\perp,0} := span\{l_1,\, l_2,\, l_3,\, q,\, c\}$;

from lemma 6.2.4 we easily deduce that

- all the quadratic harmonics are in $L^{\perp,4}$;
- all the cubic harmonics are in $L^{\perp,6}$.

By induction, we may easily conclude that,

- for $k \geq 4$, all the degree-k harmonics are in $L^{\perp,6+(k-3)(k+5)}$.

Indeed, by (6.3), $\{l \circ l\rho,\, Re(x_1 + ix_2)^{k-1}\}$ is a degree-k harmonic polynomial. By induction hypothesis $\mathcal{P}_{k-1} \subseteq L^{\perp,6+(k-4)(k+4)}$ and from lemma 6.2.4 we conclude that $\mathcal{P}_k$ is contained in $L^{\perp,6+(k-4)(k+4)+1+2k}$, i.e., $\mathcal{P}_k \subseteq L^{\perp,6+(k-3)(k+5)}$. □

Remark 6.2.1. *Let b_1, b_2 be two spherical harmonics of the same degree k. Then the sum of Poisson brackets $\{\Delta^{-1}b_1,\, b_2\} + \{\Delta^{-1}b_2,\, b_1\}$ may be written as $[k(k+1)]^{-1}(\{b_1,\, b_2\} + \{b_2,\, b_1\})$ and by the skew-symmetry of the Poisson bracket we have that $\{\Delta^{-1}b_1,\, b_2\} + \{\Delta^{-1}b_2,\, b_1\}$ vanishes. That is why we need to have all linear spherical harmonics and one degree three spherical harmonic in the set h. On the other side the sum of Poisson brackets $\{\Delta^{-1}a_n,\, a_m\} + \{\Delta^{-1}a_m,\, a_n\}$ for two spherical harmonics a_n, a_m of different (positive) degrees n and m equals $\left([n(n+1)]^{-1} - [m(m+1)]^{-1}\right)\{a_n,\, a_m\}$, i.e., the sum is collinear to $\{a_n,\, a_m\}$ with the coefficient $[n(n + 1)]^{-1} - [m(m + 1)]^{-1} \neq 0$.*

We did not consider any degree zero polynomial on the set h because, since the Sphere is compact and boundaryless, the study must be done in the space orthogonal to constants (degree zero polynomials).

Corollary 6.2.5. *Setting* $h = \left\{-x_1,\, -x_2,\, -x_3,\, 3x_3^2 - 1,\, -15x_3^3 + 9x_3\right\}$ *we obtain the following l-saturating set of vectors fields:*

$$\left(\nabla^{\perp}\cdot\right)^{-1} h = \left\{\frac{1}{2}\vec{e}_1,\, \frac{1}{2}\vec{e}_2,\, \frac{1}{2}\vec{e}_3,\, -x_3\vec{e}_3,\, \frac{3}{4}(5x_3^2 - 1)\vec{e}_3\right\},$$

where $\vec{e}_i$ *is the vector field generating rotation around the axis* $e_i = [\delta_{1i},\, \delta_{2i},\, \delta_{3i}]^{\top}$ *with constant angular velocity* 1 *(and with direction* $x \wedge e_i$*).*

Proof. Consider the chart $(u,\, v) \mapsto \left(u,\, v,\, \sqrt{1 - u^2 - v^2}\right)$ parameterizing the piece of the Sphere corresponding to $x_3 > 0$. The vectors $\partial_u,\, \partial_v$ on the Sphere are given respectively by $\left(1,\, 0,\, \frac{-u}{x_3}\right)$ and $\left(0,\, 1,\, \frac{-v}{x_3}\right)$ so; the metric tensor on the Sphere (induced by the Euclidean one of $\mathbb{R}^3$) is

$$\begin{aligned}\left(1 + \frac{u^2}{x_3^2}\right) du \otimes du &+ \frac{uv}{x_3^2}(du \otimes dv + dv \otimes du) + \left(1 + \frac{v^2}{x_3^2}\right) du \otimes du \\ &= \left(\frac{1 - v^2}{x_3^2}\right) du \otimes du + \frac{uv}{x_3^2}(du \otimes dv + dv \otimes du) + \left(\frac{1 - u^2}{x_3^2}\right) du \otimes du.\end{aligned}$$

The elements of h are eigenfunctions of the spherical Laplacean; the linear are associated with the eigenvalue $1(1+1) = 2$, the quadratic with $2(2+1) = 6$ and the cubic with $3(3+1) = 12$. The set of stream functions $\Delta^{-1}h$ associated to h is

$$s = \left\{-\frac{1}{2}x_1,\, -\frac{1}{2}x_2,\, -\frac{1}{2}x_3,\, -\frac{1}{6}\left(1 - 3x_3^2\right),\, -\frac{1}{12}\left(15x_3^3 - 9x_3\right)\right\}.$$

Thus to obtain the set of vector fields, on the fixed chart, we have to compute $-\nabla^{\perp}s$. Consider first the vorticity field $-x_3$; for $\frac{1}{2}\nabla^{\perp}x_3$ (the formula may be found in chapter 5) we obtain $\frac{1}{2}x_3\left(\frac{v}{x_3}\partial_u - \frac{u}{x_3}\partial_v\right) = \frac{1}{2}(v\partial_u - u\partial_v) = \frac{1}{2}\vec{e}_3$.

On the symmetric piece $x_3 < 0$, with chart $(u,\, v) \mapsto \left(u,\, v,\, -\sqrt{1 - u^2 - v^2}\right)$ the area element, as we have seen before is $-\frac{1}{x_3}dv \wedge du$. Similarly, in this piece, we obtain $\frac{1}{2}\nabla^{\perp}x_3 = \frac{1}{2}\vec{e}_3$ (we put $x^1 = v$ and $x^2 = u$ and make the computations accordingly with the formula in example 5.7.1).

Therefore, extending these vector field to the line $x_3 = 0$, we obtain

$$\left(\nabla^{\perp}\cdot\right)^{-1}(-x_3) = \frac{1}{2}\nabla^{\perp}x_3 = \frac{1}{2}\vec{e}_3, \quad \text{on} \quad \mathbb{S}^2.$$

Similarly

$$\left(\nabla^{\perp}\cdot\right)^{-1}(-x_1) = \frac{1}{2}\nabla^{\perp}x_1 = \frac{1}{2}\vec{e}_1, \quad \text{and} \quad \left(\nabla^{\perp}\cdot\right)^{-1}(-x_2) = \frac{1}{2}\nabla^{\perp}x_2 = \frac{1}{2}\vec{e}_2, \quad \text{on} \quad \mathbb{S}^2.$$

For $-\frac{1}{6}\nabla^{\perp}\left(3x_3^2 - 1\right)$, on $x_3 > 0$, we obtain $-\frac{1}{6}6x_3\nabla^{\perp}x_3 = -x_3\vec{e}_3$. Similarly for the piece $x_3 < 0$. Therefore

$$\left(\nabla^{\perp}\cdot\right)^{-1}(3x_3^2 - 1) = \frac{1}{6}\nabla^{\perp}(1 - 3x_3^2) = -x_3\vec{e}_3, \quad \text{on} \quad \mathbb{S}^2;$$

each point rotates around e_3 but now the angular velocity depends on x_3 On the line $x_3 = 0$ the sense of rotation changes.

Finally for vector field $-\frac{1}{12}\nabla^\perp\big(-15x_3^3+9x_3\big)$ associated with the cubic spherical harmonic we obtain $\frac{9}{12}\big(5x_3^2\nabla^\perp x_3 - \nabla^\perp x_3\big) = \frac{3}{4}\big(5x_3^2-1\big)\vec{e}_3$. Similarly for the piece $x_3 < 0$. Therefore

$$\Big(\nabla^\perp\cdot\Big)^{-1}\big(-15x_3^3+9x_3\big) = \frac{1}{12}\nabla^\perp\big(15x_3^3-9x_3\big) = \frac{3}{4}\big(5x_3^2-1\big)\vec{e}_3, \quad \text{on} \quad \mathbb{S}^2;$$

the vector field changes sense of rotation twice: at the lines $x_3 = \pm\frac{\sqrt{5}}{5}$. □

6.3 The Rectangle under Navier boundary conditions

This example has been studied in [46] under Lions boundary conditions: let R be two dimensional rectangle $R = (0,\, a)\times(0,\, b)$. From [46] we know that the set $g = \{W_n \mid n \in \mathcal{K}^1\}$ is V-saturating, where $\mathcal{K}^1 := \{(n_1,\, n_2) \in \mathbb{N}_0^2 \mid n_1,\, n_2 \le 3\} \setminus \{(3,3)\}$ and

$$W_k := \begin{pmatrix} \frac{-k_2\pi}{b}\sin\big(\frac{k_1\pi x_1}{a}\big)\cos\big(\frac{k_2\pi x_2}{b}\big) \\ \frac{k_1\pi}{a}\cos\big(\frac{k_1\pi x_1}{a}\big)\sin\big(\frac{k_2\pi x_2}{b}\big) \end{pmatrix}, \qquad k \in \mathbb{N}_0^2.$$

are eigenfunctions of the Laplace-de Rham operator with $\Delta W_k = \bar{k}W_k$ with $\bar{k} := \pi^2\left(\frac{k_1^2}{a^2}+\frac{k_2^2}{b^2}\right)$.

For any eigenfunction we have $BW_k = P^\nabla[(W_k\cdot\nabla)W_k] = 0$.

Here we check that the set g is also l-saturating.[3]

In the present case we consider the evolution of the equation in the Sobolev subspaces

$$\begin{aligned} H &:= \{u \in (L^2(R))^2 \mid \nabla\cdot u = 0,\ u\cdot\mathbf{n} = 0 \,\text{on}\, \partial R\}; \\ V &:= \{u \in (H^1(R))^2 \mid \nabla\cdot u = 0,\ u\cdot\mathbf{n} = 0 \,\text{on}\, \partial R\}; \\ D(A) &:= \{u \in (H^2(R))^2 \mid \nabla\cdot u = 0,\ (u\cdot\mathbf{n} = 0 \wedge \nabla^\perp\cdot u = 0)\,\text{on}\, \partial R\}; \end{aligned}$$

where $A = \Delta$.

For any $j \in \mathbb{N}_0$ put

$$g^{j-1} = \big\{W_n \mid n \in \mathcal{K}^j\big\}, \quad \mathcal{K}^j := \big\{(n_1,\, n_2) \in \mathbb{N}_0^2 \mid n_1,\, n_2 \le j+2\big\} \setminus \{(j+2, j+2)\}.$$

We shall prove that the l-saturating sequence $(L^m)_{m\in\mathbb{N}}$ given by the definition of l-saturating set satisfy $span(g^m) \subseteq L^{m+1}$. This implies in particular that g is l-saturating.

For $-Bu$, with $u = \sum_{k\in\mathbb{N}_0^2} u_k W_k$, after some computations,[4] we can arrive to the following

[3] The rectangle is not a C^∞ domain and for domains not regular enough we may loose regularity at each step in the construction of the sequence in the definition of l-saturating set. Anyway in the particular case of the rectangle we know explicitly the eigenfunctions and the expansion of Bu in a Fourier series; we are able to manage.

[4] The computation is direct but, since it is quite long, we do not present it here. Detais may be found in [48].

expression

$$
\begin{aligned}
-Bu &= -P^{\nabla}[(u\cdot\nabla)u] \\
&= \sum_{\substack{m,n\in\mathbb{N}_0^2\\ m<n}} \frac{\pi^2}{ab}\frac{u_m u_n (m\wedge n)}{\overline{(n(++)m)^+}}(\bar{n}-\bar{m})W_{(n(++)m)^+} \\
&\quad + \sum_{\substack{m,n\in\mathbb{N}_0^2\\ m<n}} -\frac{\pi^2}{ab}\frac{u_m u_n (m\vee n)}{\overline{(n(+-)m)^+}}(\bar{n}-\bar{m})\operatorname{sign}(n_2-m_2)W_{(n(+-)m)^+} \\
&\quad + \sum_{\substack{m,n\in\mathbb{N}_0^2\\ m<n}} \frac{\pi^2}{ab}\frac{u_m u_n (m\vee n)}{\overline{(n(-+)m)^+}}(\bar{n}-\bar{m})\operatorname{sign}(n_1-m_1)W_{(n(-+)m)^+} \\
&\quad + \sum_{\substack{m,n\in\mathbb{N}_0^2\\ m<n}} -\frac{\pi^2}{ab}\frac{u_m u_n (m\wedge n)}{\overline{(n(--)m)^+}}(\bar{n}-\bar{m})\operatorname{sign}(n_1-m_1)\operatorname{sign}(n_2-m_2)W_{(n(--)m)^+}. \qquad (6.4)
\end{aligned}
$$

Here $(n(\alpha\beta)m)^+ := (|n_1\alpha m_1|,\, |n_2\beta m_2|)$, $\alpha,\,\beta\in\{+,\,-\}$; $m\wedge n = m_1n_2 - m_2n_1$ and $m\vee n = m_1n_2 + m_2n_1$.

Select the subset

$$F_{S_1} := \{\delta_{m,n} = -B(W_m,\,W_n) - B(W_n,\,W_m) \mid (m,\,n)\in S_1\subset(\mathcal{K}^1)^2\}$$

from L^1, where

$$
\begin{aligned}
S_1 = \{&((1,2),(2,1));\ ((1,1),(2,3));\ ((1,2),(2,2)); \\
&((1,1),(3,2));\ ((2,1),(2,2));\ ((1,1),(1,3));\ ((1,1),(3,1))\}.
\end{aligned}
$$

The vectors of this family are precisely (see the annex at the end of this section):

$$
\begin{aligned}
\delta_{(1,2),(2,1)} &= \frac{9\pi^2(b^2-a^2)}{4ab(a^2+b^2)}W_{(1,1)} + \frac{15\pi^2(a^2-b^2)}{4ab(9a^2+b^2)}W_{(1,3)} \\
&\quad + \frac{15\pi^2(a^2-b^2)}{4ab(a^2+9b^2)}W_{(3,1)} + \frac{\pi^2(b^2-a^2)}{4ab(a^2+b^2)}W_{(3,3)} \\
\delta_{(1,1),(2,3)} &= \frac{\pi^2(3b^2+8a^2)}{4ab(4a^2+b^2)}W_{(1,2)} - \frac{5\pi^2(8a^2+3b^2)}{4ab(16a^2+b^2)}W_{(1,4)} \\
&\quad + \frac{5\pi^2(8a^2+3b^2)}{4ab(4a^2+9b^2)}W_{(3,2)} - \frac{\pi^2(3b^2+8a^2)}{4ab(16a^2+9b^2)}W_{(3,4)}
\end{aligned}
$$

$$
\begin{aligned}
\delta_{(1,2),(2,2)} &= -\frac{9b\pi^2}{2a(16a^2+b^2)}W_{(1,4)} + \frac{3b\pi^2}{2a(16a^2+9b^2)}W_{(3,4)} \\
\delta_{(1,1),(3,2)} &= -\frac{\pi^2(8b^2+3a^2)}{4ab(a^2+4b^2)}W_{(2,1)} - \frac{5\pi^2(3a^2+8b^2)}{4ab(9a^2+4b^2)}W_{(2,3)} \\
&\quad + \frac{5\pi^2(3a^2+8b^2)}{4ab(a^2+16b^2)}W_{(4,1)} + \frac{\pi^2(8b^2+3a^2)}{4ab(9a^2+16b^2)}W_{(4,3)}
\end{aligned}
$$

$$
\begin{aligned}
\delta_{(2,1),(2,2)} &= -\frac{9a\pi^2}{2b(16b^2+a^2)}W_{(4,1)} - \frac{3a\pi^2}{2b(16b^2+9a^2)}W_{(4,3)}\\
\delta_{(1,1),(1,3)} &= \frac{2a\pi^2}{b(b^2+a^2)}W_{(2,2)} - \frac{a\pi^2}{b(b^2+4a^2)}W_{(2,4)}\\
\delta_{(1,1),(3,1)} &= -\frac{2b\pi^2}{a(b^2+a^2)}W_{(2,2)} + \frac{b\pi^2}{a(a^2+4b^2)}W_{(4,2)}.
\end{aligned}
$$

Projecting the vectors in this subfamily on the space $span\{W_k \mid k \in \mathcal{K}^2 \setminus \mathcal{K}^1\}$ we obtain

$$
\Pi_1\delta_{(1,2),(2,1)} = \frac{\pi^2(b^2-a^2)}{4ab(a^2+b^2)}W_{(3,3)} \tag{6.5}
$$

$$
\Pi_1\delta_{(1,1),(2,3)} = -\frac{5\pi^2(8a^2+3b^2)}{4ab(16a^2+b^2)}W_{(1,4)} - \frac{\pi^2(3b^2+8a^2)}{4ab(16a^2+9b^2)}W_{(3,4)}
$$

$$
\begin{aligned}
\Pi_1\delta_{(1,2),(2,2)} &= -\frac{9b\pi^2}{2a(16a^2+b^2)}W_{(1,4)} + \frac{3b\pi^2}{2a(16a^2+9b^2)}W_{(3,4)}\\
\Pi_1\delta_{(1,1),(3,2)} &= \frac{5\pi^2(3a^2+8b^2)}{4ab(a^2+16b^2)}W_{(4,1)} + \frac{\pi^2(8b^2+3a^2)}{4ab(9a^2+16b^2)}W_{(4,3)}
\end{aligned}
$$

$$
\begin{aligned}
\Pi_1\delta_{(2,1),(2,2)} &= -\frac{9a\pi^2}{2b(16b^2+a^2)}W_{(4,1)} - \frac{3a\pi^2}{2b(16b^2+9a^2)}W_{(4,3)}\\
\Pi_1\delta_{(1,1),(1,3)} &= -\frac{a\pi^2}{b(b^2+4a^2)}W_{(2,4)}\\
\Pi_1\delta_{(1,1),(3,1)} &= \frac{b\pi^2}{a(a^2+4b^2)}W_{(4,2)}.
\end{aligned}
$$

In the case $a \neq b$ these projections are linearly independent so, the 15 vectors in the family $\{W_k \mid k \in \mathcal{K}^1\} \cup F_{S_1}$ are linearly independent and they span $span(g^1)$. Therefore we have the inclusion $span(g^1) \subseteq L^1$.

In the case $a = b$ we may extract from L^1 the vectors in

$$
F_{S_{1a}} := \{\delta_{m,n} = -B(W_m, W_n) - B(W_n, W_m) \mid (m, n) \in (S_1 \setminus \{((1,2),(2,1))\})\}
$$

and obtain the previous family of vectors but the vector in (6.5), i.e., we have the inclusion $span(g^1 \setminus \{W_{(3,3)}\}) \subseteq L^1$.

Then from L^2 we extract the family

$$\begin{aligned}
\delta_{(1,1),(2,4)} =& \frac{3\pi^2(5a^2+b^2)}{2ab(9a^2+b^2)}W_{(1,3)} - \frac{9\pi^2(5a^2+b^2)}{2ab(25a^2-b^2)}W_{(1,5)} \\
&+\frac{\pi^2(5a^2+b^2)}{2ab(a^2+b^2)}W_{(3,3)} - \frac{3\pi^2(5a^2+b^2)}{2ab(25a^2+9b^2)}W_{(3,5)} \\
\delta_{(1,2),(2,3)} =& -\frac{\pi^2(5a^2+3b^2)}{4ab(a^2+b^2)}W_{(1,1)} - \frac{7\pi^2(5a^2+3b^2)}{4ab(25a^2+b^2)}W_{(1,5)} \\
&+\frac{7\pi^2(5a^2+3b^2)}{4ab(a^2+9b^2)}W_{(3,1)} + \frac{\pi^2(5a^2+3b^2)}{4ab(25a^2+9b^2)}W_{(3,5)} \\
\delta_{(1,4),(2,1)} =& \frac{21\pi^2(b^2-5a^2)}{4ab(9a^2+b^2)}W_{(1,3)} + \frac{27\pi^2(5a^2-b^2)}{4ab(25a^2+b^2)}W_{(1,5)} \\
&+\frac{3\pi^2(5a^2-b^2)}{4ab(a^2+b^2)}W_{(3,3)} + \frac{21\pi^2(b^2-5a^2)}{4ab(25a^2+9b^2)}W_{(3,5)}.
\end{aligned}$$

Projecting in $span\{W_k \mid k \in \mathcal{K}^3 \setminus (\mathcal{K}^2 \setminus \{(3,3)\})\}$ we obtain

$$\begin{aligned}
\Pi\delta_{(1,1),(2,4)} =& -\frac{9\pi^2(5a^2+b^2)}{2ab(25a^2-b^2)}W_{(1,5)} \\
&+\frac{\pi^2(5a^2+b^2)}{2ab(a^2+b^2)}W_{(3,3)} - \frac{3\pi^2(5a^2+b^2)}{2ab(25a^2+9b^2)}W_{(3,5)} \\
\Pi\delta_{(1,2),(2,3)} =& -\frac{7\pi^2(5a^2+3b^2)}{4ab(25a^2+b^2)}W_{(1,5)} + \frac{\pi^2(5a^2+3b^2)}{4ab(25a^2+9b^2)}W_{(3,5)} \\
\Pi\delta_{(1,4),(2,1)} =& \frac{27\pi^2(5a^2-b^2)}{4ab(25a^2+b^2)}W_{(1,5)} \\
&+\frac{3\pi^2(5a^2-b^2)}{4ab(a^2+b^2)}W_{(3,3)} + \frac{21\pi^2(b^2-5a^2)}{4ab(25a^2+9b^2)}W_{(3,5)}.
\end{aligned}$$

Since we are working in the square, $a = b$, no one of the coefficients appearing in the last expressions vanish.

For simplicity writing (6.4) in the form

$$\begin{aligned}
-Bu =& -P^{\nabla}[(u\cdot\nabla)u] \\
=& \sum_{\substack{m,n\in\mathbb{N}_0^2 \\ m<n}} u_m u_n C^{++}_{m,n} W_{(n(++)m)^+} + \sum_{\substack{m,n\in\mathbb{N}_0^2 \\ m<n}} u_m u_n C^{+-}_{m,n} W_{(n(+-)m)^+} \\
&+ \sum_{\substack{m,n\in\mathbb{N}_0^2 \\ m<n}} u_m u_n C^{-+}_{m,n} W_{(n(-+)m)^+} + \sum_{\substack{m,n\in\mathbb{N}_0^2 \\ m<n}} u_m u_n C^{--}_{m,n} W_{(n(--)m)^+};
\end{aligned}$$

where the coefficients $C^{++}_{m,n}$, $C^{+-}_{m,n}$, $C^{-+}_{m,n}$ and $C^{--}_{m,n}$ agree with (6.4). It turns out that

$$\begin{aligned}
&\det\begin{pmatrix} C^{-+}_{(1,1),(2,4)} & C^{+-}_{(1,1),(2,4)} & C^{++}_{(1,1),(2,4)} \\ C^{-+}_{(1,2),(2,3)} & 0 & C^{++}_{(1,2),(2,3)} \\ C^{-+}_{(1,4),(2,1)} & C^{+-}_{(1,4),(2,1)} & C^{++}_{(1,4),(2,1)} \end{pmatrix} \\
=& -\frac{15\pi^2(125a^6+75a^4b^2-5a^2b^4-3b^6)}{16a^3b^3(a^2+b^2)(25a^2+b^2)(25a^2+9b^2)} = -\frac{\pi^2 2880a^6}{28288a^{12}} = -\frac{\pi^2 45}{442a^6} \neq 0;
\end{aligned}$$

thus the last three vectors are linearly independent and if we join them to the vectors in $\{W_k \mid k \in \mathcal{K}^2 \setminus \{(3,3)\}\} \subseteq L^1$, we obtain a family of 4^2+1 linearly independent vectors spanning the space $\{W_k \mid k \in \mathcal{K}^2 \cup \{(1,5),(3,5)\}\}$. In particular we have that $span(g^1) \subseteq L^2$.

In both cases $a = b$ or $a \neq b$ we have $span(g^1) \subseteq L^2$.

Now we prove that if $g^j \subseteq L^{j+1}$, then $g^{j+1} \subseteq L^{j+2}$ what gives the result.

We consider two cases "j even" and "j odd". We consider also $j \geq 1$, because the case $j = 0$ has already been proven.

- j even: In this case

$$\mathcal{K}^{j+1} := \big\{(n_1,\, n_2) \in \mathbb{N}_0^2 \mid n_1,\, n_2 \leq j+3\big\} \setminus \{(j+3, j+3)\}.$$

can be written as

$$\mathcal{K}^{j+1} = \big\{(n_1,\, n_2) \in \mathbb{N}_0^2 \mid n_1,\, n_2 \leq 2p+1\big\} \setminus \{(2p+1, 2p+1)\},$$

setting $p = \frac{j+2}{2}$. Then $p \geq 2$.
As we did before in the case "$j = 0 \to j = 1$", from L^{j+2}, we extract a subfamily

$$F_{S_{j+1}} := \{\delta_{m,n} = -B(W_m,\, W_n) - B(W_n,\, W_m) \mid (m,\, n) \in S_{j+1} \subset \mathcal{K}^1 \times \mathcal{K}^{j+1}\}$$

where now the "selection" is

$$\begin{aligned}
S_{j+1} = \{&((1,2),(2p,2p-1))\} \\
&\cup \{((1,1),(2z,2p+1)) \mid z = 1,\, \ldots,\, p\} \cup \{((1,3)(2,2p-1))\} \\
&\cup \{((1,1),(2p+1,2z)) \mid z = 1,\, \ldots,\, p\} \cup \{((3,1),(2p-1,2))\} \\
&\cup \{((s,1),(s,2p+1)) \mid s = 1,\, \ldots,\, \max\{p,\, 3\}\} \\
&\cup \{((3,1),(2s-3,2p+1)) \mid s = 4,\, \ldots,\, p;\, p \geq 4\} \\
&\cup \{((1,s),(2p+1,s)) \mid s = 1,\, \ldots,\, \max\{p,\, 3\}\} \\
&\cup \{((1,3),(2p+1,2s-3)) \mid s = 4,\, \ldots,\, p;\, p \geq 4\}.
\end{aligned}$$

If we write explicitly the vectors of $F_{S_{j+1}}$ we obtain quite long expressions, for example we have that $C^{--}_{(1,2),(2p,2p-1)}$ equals

$$-\frac{\big(a^2(-3+2p) + b^2(-1+2p)\big)(\pi + 2p\pi)^2 \text{sign}(-3+2p)\text{sign}(-1+2p)}{4ab(b^2|1-2p|^2 + a^2|3-2p|^2)};$$

so, here we will not write those vectors explicitly, instead we write them as

$$\begin{aligned}
\delta_{(1,2),(2p,2p-1)} =& C^{--}_{(1,2),(2p,2p-1)} W_{(2p-1,2p-3)} + C^{-+}_{(1,2),(2p,2p-1)} W_{(2p-1,2p+1)} \\
&+ C^{+-}_{(1,2),(2p,2p-1)} W_{(2p+1,2p-3)} + C^{++}_{(1,2),(2p,2p-1)} W_{(2p+1,2p+1)}; \\
\delta_{(1,1),(2z,2p+1)} =& C^{--}_{(1,1),(2z,2p+1)} W_{(2z-1,2p)} + C^{-+}_{(1,1),(2z,2p+1)} W_{(2z-1,2(p+1))} \\
&+ C^{+-}_{(1,1),(2z,2p+1)} W_{(2z+1,2p)} + C^{++}_{(1,1),(2z,2p+1)} W_{(2z+1,2(p+1))} \\
& z = 1,\, \ldots,\, p;
\end{aligned}$$

$$\begin{aligned}
\delta_{(1,3),(2,2p-1)} =& C^{--}_{(1,3),(2,2p-1)} W_{(1,2p-4)} + C^{-+}_{(1,3),(2,2p-1)} W_{(1,2(p+1))} \\
&+ C^{+-}_{(1,3),(2,2p-1)} W_{(3,2p-4)} + C^{++}_{(1,3),(2,2p-1)} W_{(3,2(p+1))}; \\
\delta_{(1,1),(2p+1,2z)} =& C^{--}_{(1,1),(2p+1,2z)} W_{(2p,2z-1)} + C^{-+}_{(1,1),(2p+1,2z)} W_{(2p,2z+1)} \\
&+ C^{+-}_{(1,1),(2p+1,2z)} W_{(2(p+1),2z-1)} + C^{++}_{(1,1),(2p+1,2z)} W_{(2(p+1),2z+1)} \\
&z = 1, \ldots, p;
\end{aligned}$$

$$\begin{aligned}
\delta_{(3,1),(2p-1,2)} =& C^{--}_{(3,1),(2p-1,2)} W_{(2p-4,1)} + C^{-+}_{(3,1),(2p-1,2)} W_{(2p-4,3)} \\
&+ C^{+-}_{(3,1),(2p-1,2)} W_{(2(p+1),1)} + C^{++}_{(3,1),(2p-1,2)} W_{(2(p+1),3)}; \\
\delta_{(s,1),(s,2p+1)} =& C^{+-}_{(s,1),(s,2p+1)} W_{(2s,2p)} + C^{++}_{(s,1),(s,2p+1)} W_{(2s,2(p+1))} \\
&s = 1, \ldots, \max\{p, 3\}; \\
\delta_{(3,1),(2s-3,2p+1)} =& C^{--}_{(3,1),(2s-3,2p+1)} W_{(2s-6,2p)} + C^{-+}_{(3,1),(2s-3,2p+1)} W_{(2s-6,2p+2)} \\
&+ C^{+-}_{(3,1),(2s-3,2p+1)} W_{(2s,2p)} + C^{++}_{(3,1),(2s-3,2p+1)} W_{(2s,2p+2)}; \\
&s = 4, \ldots, p; \ p \geq 4;
\end{aligned}$$

$$\begin{aligned}
\delta_{(1,s),(2p+1,s)} =& C^{-+}_{(1,s),(2p+1,s)} W_{(2p,2s)} + C^{++}_{(1,s),(2p+1,s)} W_{(2(p+1),2s)} \\
&s = 1, \ldots, \max\{p, 3\}; \\
\delta_{(1,3),(2p+1,2s-3)} =& C^{--}_{(1,3),(2p+1,2s-3)} W_{(2p,2s-6)} + C^{-+}_{(1,3),(2p+1,2s-3)} W_{(2p,2s)} \\
&+ C^{+-}_{(1,3),(2p+1,2s-3)} W_{(2p+2,2s-6)} + C^{++}_{(1,3),(2p+1,2s-3)} W_{(2p+2,2s)}; \\
&s = 4, \ldots, p; \ p \geq 4.
\end{aligned}$$

And, projecting them onto the space $span\{e_k \mid k \in \mathcal{K}^{j+2} \setminus \mathcal{K}^{j+1}\}$ we arrive to the family $\Pi_{j+1} F_{S_{j+1}}$ whose elements are

$$\begin{aligned}
\Pi_{j+1}\delta_{(1,2),(2p,2p-1)} =& C^{++}_{(1,2),(2p,2p-1)} W_{(2p+1,2p+1)}; \\
\Pi_{j+1}\delta_{(1,1),(2z,2p+1)} =& C^{-+}_{(1,1),(2z,2p+1)} W_{(2z-1,2(p+1))} + C^{++}_{(1,1),(2z,2p+1)} W_{(2z+1,2(p+1))} \\
&z = 1, \ldots, p;
\end{aligned} \tag{6.6}$$

$$\begin{aligned}
\Pi_{j+1}\delta_{(1,3),(2,2p-1)} =& C^{-+}_{(1,3),(2,2p-1)} W_{(1,2(p+1))} + C^{++}_{(1,3),(2,2p-1)} W_{(3,2(p+1))}; \\
\Pi_{j+1}\delta_{(1,1),(2p+1,2z)} =& C^{+-}_{(1,1),(2p+1,2z)} W_{(2(p+1),2z-1)} + C^{++}_{(1,1),(2p+1,2z)} W_{(2(p+1),2z+1)} \\
&z = 1, \ldots, p;
\end{aligned} \tag{6.7}$$

$$\begin{aligned}
\Pi_{j+1}\delta_{(3,1),(2p-1,2)} =& C^{-+}_{(3,1),(2p-1,2)} W_{(2p-4,3)} + C^{++}_{(3,1),(2p-1,2)} W_{(2(p+1),3)}; \\
\Pi_{j+1}\delta_{(s,1),(s,2p+1)} =& C^{++}_{(s,1),(s,2p+1)} W_{(2s,2(p+1))}, \qquad s = 1, \ldots, \max\{p, 3\}; \\
\Pi_{j+1}\delta_{(3,1),(2s-3,2p+1)} =& C^{-+}_{(3,1),(2s-3,2p+1)} W_{(2s-6,2p+2)} + C^{++}_{(3,1),(2s-3,2p+1)} W_{(2s,2p+2)}; \\
&s = 4, \ldots, p; \ p \geq 4;
\end{aligned}$$

$$\begin{aligned}\Pi_{j+1}\delta_{(1,s),(2p+1,s)} =& C^{++}_{(1,s),(2p+1,s)} W_{(2(p+1),2s)}, \qquad s = 1, \ldots, \max\{p, 3\};\\ \Pi_{j+1}\delta_{(1,3),(2p+1,2s-3)} =& C^{+-}_{(1,3),(2p+1,2s-3)} W_{(2p+2,2s-6)} + C^{++}_{(1,3),(2p+1,2s-3)} W_{(2p+2,2s)};\\ & s = 4, \ldots, p;\ p \geq 4.\end{aligned}$$

No one of the coefficients appearing in these expressions vanishes because all pairs (m, n) satisfy $m < n \wedge (m_1 = n_1 \vee n_2 \geq m_2)$, which implies that $\bar{n} \neq \bar{m}$, and because no one of the following expressions vanish

$$\begin{aligned}(1,2) \wedge (2p, 2p-1) &= -1 - 2p;\\ (1,1) \wedge (2z, 2p+1) &= 1 + 2p - 2z, \quad z = 1, \ldots, p;\\ (1,3) \wedge (2, 2p-1) &= -7 + 2p;\\ (1,1) \wedge (2p+1, 2z) &= -1 - 2p + 2z, \quad z = 1, \ldots, p;\\ (3,1) \wedge (2p-1, 2) &= 7 - 2p;\\ (s,1) \wedge (s, 2p+1) &= 2ps, \quad s = 1, \ldots, \max\{p, 3\};\\ (3,1) \wedge (2s-3, 2p+1) &= 6 + 6p - 2s, \quad s = 4, \ldots, p;\ p \geq 4;\\ (1,s) \wedge (2p+1, s) &= -2ps, \quad s = 1, \ldots, \max\{p, 3\};\\ (1,3) \wedge (2p+1, 2s-3) &= -6 - 6p + 2s, \quad s = 4, \ldots, p;\ p \geq 4.\end{aligned}$$

Hence we can see that these vectors are linearly independent. Indeed it suffices to prove that:

- The vectors $\Pi_{j+1}\delta_{(1,1),(2,2p+1)}$, $(z = 1$ in (6.6)) and $\Pi_{j+1}\delta_{(1,3),(2,2p-1)}$ are linearly independent; and
- The vectors $\Pi_{j+1}\delta_{(1,1),(2p+1,2)}$, $(z = 1$ in (6.7)) and $\Pi_{j+1}\delta_{(3,1),(2p-1,2)}$ are linearly independent;

But that, since p is an integer greater than 1, comes from

$$\begin{aligned}\frac{C^{-+}_{(1,1),(2,2p+1)}}{C^{-+}_{(1,3),(2,2p-1)}} =& \frac{(3+2p)(3b^2 + 4a^2 p(1+p))}{(5+2p)(3b^2 + 4a^2(-2+p)(1+p))}\\ \neq& \frac{(-1+2p)(3b^2 + 4a^2 p(1+p))}{(-7+2p)(3b^2 + 4a^2(-2+p)(1+p))} = \frac{C^{++}_{(1,1),(2,2p+1)}}{C^{++}_{(1,3),(2,2p-1)}};\end{aligned}$$

and

$$\begin{aligned}\frac{C^{+-}_{(1,1),(2p+1,2)}}{C^{+-}_{(3,1),(2p-1,2)}} =& \frac{(3+2p)(3a^2 + 4b^2 p(1+p))}{(5+2p)(3a^2 + 4b^2(-2+p)(1+p))}\\ \neq& \frac{(-1+2p)(3a^2 + 4b^2 p(1+p))}{(-7+2p)(3a^2 + 4b^2(-2+p)(1+p))} = \frac{C^{++}_{(1,1),(2p+1,2)}}{C^{++}_{(3,1),(2p-1,2)}}.\end{aligned}$$

Then the $(2(p+1))^2 - 1$ vectors in $F_{S_{j+1}} \cup \{W_k \mid k \in \mathcal{K}^{j+1}\}$ are linearly independent and span

$$\begin{aligned}& span\{W_k \mid 1 \leq k_1, k_2 \leq 2(p+1)\} \setminus \{(2(p+1), 2(p+1))\}\\ =\ & span(g^{j+1}).\end{aligned}$$

Therefore $span(g^{j+1}) \subseteq L^{j+2}$.

- j odd: In this case

$$\mathcal{K}^{j+1} := \left\{(n_1,\, n_2) \in \mathbb{N}_0^2 \mid n_1,\, n_2 \leq j+3\right\} \setminus \{(j+3, j+3)\}.$$

can be written as

$$\mathcal{K}^{j+1} = \left\{(n_1,\, n_2) \in \mathbb{N}_0^2 \mid n_1,\, n_2 \leq 2p\right\} \setminus \{(2p, 2p)\},$$

setting $p = \frac{j+3}{2}$. Then $p \geq 2$.
We extract the subfamily

$$F_{S_{j+1}} := \{\delta_{m,n} = -B(W_n,\, W_m) - B(W_m,\, W_n) \mid \; (m,\, n) \in S_{j+1} \subset \mathcal{K}^1 \times \mathcal{K}^{j+1}\}$$

from L^{j+2} where now the "selection" is

$$\begin{aligned} S_{j+1} = \{&((1,2),(2p-1,2p-2))\} \\ &\cup \{((1,1),(2z-1,2p)) \mid \; z = 2,\, \dots,\, p\} \cup \{((1,2)(3,2p-1))\} \\ &\cup \{((1,1),(2p,2z-1)) \mid \; z = 2,\, \dots,\, p\} \cup \{((2,1),(2p-1,3))\} \\ &\cup \{((1,1),(2s,2p)) \mid \; s = 1,\, \dots,\, p-1\} \cup \{((1,2),(2,2p-1))\} \\ &\cup \{((1,1),(2p,2s)) \mid \; s = 1,\, \dots,\, p-1\} \cup \{((2,1),(2p-1,2))\}. \end{aligned}$$

The respective vectors are

$$\begin{aligned} \delta_{(1,2),(2p-1,2p-2)} =& C^{--}_{(1,2),(2p-1,2p-2)} W_{(2p-2,2p-4)} + C^{-+}_{(1,2),(2p-1,2p-2)} W_{(2p-2,2p)} \\ &+ C^{+-}_{(1,2),(2p-1,2p-2)} W_{(2p,2p-4)} + C^{++}_{(1,2),(2p-1,2p-2)} W_{(2p,2p)}; \\ \delta_{(1,1),(2z-1,2p)} =& C^{--}_{(1,1),(2z-1,2p)} W_{(2(z-1),2p-1)} + C^{-+}_{(1,1),(2z-1,2p)} W_{(2(z-1),2p+1)} \\ &+ C^{+-}_{(1,1),(2z-1,2p)} W_{(2z,2p-1)} + C^{++}_{(1,1),(2z-1,2p)} W_{(2z,2p+1)} \\ & z = 2,\, \dots,\, p; \end{aligned}$$

$$\begin{aligned} \delta_{(1,2)(3,2p-1)} =& C^{--}_{(1,2)(3,2p-1)} W_{(2,2p-3)} + C^{-+}_{(1,2)(3,2p-1)} W_{(2,2p+1)} \\ &+ C^{+-}_{(1,2)(3,2p-1)} W_{(4,2p-3)} + C^{++}_{(1,2)(3,2p-1)} W_{(4,2p+1)}; \\ \delta_{(1,1),(2p,2z-1)} =& C^{--}_{(1,1),(2p,2z-1)} W_{(2p-1,2(z-1))} + C^{-+}_{(1,1),(2p,2z-1)} W_{(2p-1,2z)} \\ &+ C^{+-}_{(1,1),(2p,2z-1)} W_{(2p+1,2(z-1))} + C^{++}_{(1,1),(2p,2z-1)} W_{(2p+1,2z)} \\ & z = 2,\, \dots,\, p; \end{aligned}$$

$$\begin{aligned} \delta_{(2,1),(2p-1,3)} =& C^{--}_{(2,1),(2p-1,3)} W_{(2p-3,2)} + C^{-+}_{(2,1),(2p-1,3))} W_{(2p-3,4)} \\ &+ C^{+-}_{(2,1),(2p-1,3)} W_{(2p+1,2)} + C^{++}_{(2,1),(2p-1,3)} W_{(2p+1,4)}; \\ \delta_{(1,1),(2s,2p)} =& C^{--}_{(1,1),(2s,2p)} W_{(2s-1,2p-1)} + C^{-+}_{(1,1),(2s,2p)} W_{(2s-1,2p+1)} \\ &+ C^{+-}_{(1,1),(2s,2p)} W_{(2s+1,2p-1)} + C^{++}_{(1,1),(2s,2p)} W_{(2s+1,2p+1)} \\ & s = 1,\, \dots,\, p-1; \end{aligned}$$

$$\begin{aligned}
\delta_{(1,2),(2,2p-1)} =& C^{--}_{(1,2),(2,2p-1)}W_{(1,2p-3)} + C^{-+}_{(1,2),(2,2p-1)}W_{(1,2p+1)}\\
&+C^{+-}_{(1,2),(2,2p-1)}W_{(3,2p-3)} + C^{++}_{(1,2),(2,2p-1)}W_{(3,2p+1)};\\
\delta_{(1,1),(2p,2s)} =& C^{--}_{(1,1),(2p,2s)}W_{(2p-1,2s-1)} + C^{-+}_{(1,1),(2p,2s)}W_{(2p-1,2s+1)}\\
&+C^{+-}_{(1,1),(2p,2s)}W_{(2p+1,2s-1)} + C^{++}_{(1,1),(2p,2s)}W_{(2p+1,2s+1)}\\
&s = 1,\,\dots,\,p-1;\\
\delta_{(2,1),(2p-1,2)} =& C^{--}_{(2,1),(2p-1,2)}W_{(2p-3,1)} + C^{-+}_{(2,1),(2p-1,2)}W_{(2p-3,3)}\\
&+C^{+-}_{(2,1),(2p-1,2)}W_{(2p+1,1)} + C^{++}_{(2,1),(2p-1,2)}W_{(2p+1,3)}.
\end{aligned}$$

Projecting them onto the space $span\{e_k \mid k \in \mathcal{K}^{j+2}\setminus\mathcal{K}^{j+1}\}$, we arrive to the family $\Pi_{j+1}F_{S_{j+1}}$ which elements are

$$\begin{aligned}
\Pi_{j+1}\delta_{(1,2),(2p-1,2p-2)} =& C^{++}_{(1,2),(2p-1,2p-2)}W_{(2p,2p)};\\
\Pi_{j+1}\delta_{(1,1),(2z-1,2p)} =& C^{-+}_{(1,1),(2z-1,2p)}W_{(2(z-1),2p+1)} + C^{++}_{(1,1),(2z-1,2p)}W_{(2z,2p+1)}\\
&z = 2,\,\dots,\,p;
\end{aligned} \tag{6.8}$$

$$\begin{aligned}
\Pi_{j+1}\delta_{(1,2)(3,2p-1)} =& C^{-+}_{(1,2)(3,2p-1)}W_{(2,2p+1)} + C^{++}_{(1,2)(3,2p-1)}W_{(4,2p+1)};\\
\Pi_{j+1}\delta_{(1,1),(2p,2z-1)} =& C^{+-}_{(1,1),(2p,2z-1)}W_{(2p+1,2(z-1))} + C^{++}_{(1,1),(2p,2z-1)}W_{(2p+1,2z)}\\
&z = 2,\,\dots,\,p;
\end{aligned} \tag{6.9}$$

$$\begin{aligned}
\Pi_{j+1}\delta_{(2,1),(2p-1,3)} =& C^{+-}_{(2,1),(2p-1,3)}W_{(2p+1,2)} + C^{++}_{(2,1),(2p-1,3)}W_{(2p+1,4)};\\
\Pi_{j+1}\delta_{(1,1),(2s,2p)} =& C^{-+}_{(1,1),(2s,2p)}W_{(2s-1,2p+1)} + C^{++}_{(1,1),(2s,2p)}W_{(2s+1,2p+1)}\\
&s = 1,\,\dots,\,p-1;
\end{aligned} \tag{6.10}$$

$$\begin{aligned}
\Pi_{j+1}\delta_{(1,2),(2,2p-1)} =& C^{-+}_{(1,2),(2,2p-1)}W_{(1,2p+1)} + C^{++}_{(1,2),(2,2p-1)}W_{(3,2p+1)};\\
\Pi_{j+1}\delta_{(1,1),(2p,2s)} =& C^{+-}_{(1,1),(2p,2s)}W_{(2p+1,2s-1)} + C^{++}_{(1,1),(2p,2s)}W_{(2p+1,2s+1)}\\
&s = 1,\,\dots,\,p-1; \qquad (6.11)\\
\Pi_{j+1}\delta_{(2,1),(2p-1,2)} =& C^{+-}_{(2,1),(2p-1,2)}W_{(2p+1,1)} + C^{++}_{(2,1),(2p-1,2)}W_{(2p+1,3)}.
\end{aligned}$$

No one of the coefficients appearing in these expressions vanishes because all pairs (m,n) satisfy $m < n \wedge (m_1 = n_1 \vee n_2 \geq m_2)$ and because no one of the following expressions vanish

$$\begin{aligned}
(1,2)\wedge(2p-1,2p-2) &= -2p;\\
(1,1)\wedge(2z-1,2p) &= 1+2(p-z), \quad z = 2,\,\dots,\,p;\\
(1,2)\wedge(3,2p-1) &= -7+2p;\\
(1,1)\wedge(2p,2z-1) &= -1-2(p-z), \quad z = 2,\,\dots,\,p;\\
(2,1)\wedge(2p-1,3) &= 7-2p;\\
(1,1)\wedge(2s,2p) &= 2(p-s), \quad s = 1,\,\dots,\,p-1;\\
(1,2)\wedge(2,2p-1) &= -5+2p;\\
(1,1)\wedge(2p,2s) &= 2(s-p), \quad s = 1,\,\dots,\,p-1\\
(2,1)\wedge(2p-1,2) &= 5-2p.
\end{aligned}$$

Hence we can see that these vectors are linearly independent. To see this is enough to see that:

- The vectors $\Pi_{j+1}\delta_{(1,1),(3,2p)}$, $(z = 2 \text{ in } (6.8))$ and $\Pi_{j+1}\delta_{(1,2)(3,2p-1)}$ are linearly independent;
- The vectors $\Pi_{j+1}\delta_{(1,1),(2p,3)}$, $(z = 2 \text{ in } (6.9))$ and $\Pi_{j+1}\delta_{(2,1),(2p-1,3)}$ are linearly independent;
- The vectors $\Pi_{j+1}\delta_{(1,1),(2,2p)}$, $(s = 1 \text{ in } (6.10))$ and $\Pi_{j+1}\delta_{(1,2)(2,2p-1)}$ are linearly independent;
- The vectors $\Pi_{j+1}\delta_{(1,1),(2p,2)}$, $(s = 1 \text{ in } (6.11))$ and $\Pi_{j+1}\delta_{(2,1),(2p-1,2)}$ are linearly independent;

But, since p is a natural number greater that 1, that comes from

$$\begin{aligned}\frac{C^{-+}_{(1,1),(3,2p)}}{C^{-+}_{(1,2)(3,2p-1)}} &= \frac{(3+2p)(8b^2+a^2(4p^2-1))}{(5+2p)(8b^2+a^2(2p-3)(2p+1))} \\ &\neq \frac{(2p-3)(8b^2+a^2(4p^2-1))}{(2p-7)(8b^2+a^2(2p-3)(2p+1))} = \frac{C^{++}_{(1,1),(3,2p)}}{C^{++}_{(1,2)(3,2p-1)}};\end{aligned}$$

$$\begin{aligned}\frac{C^{+-}_{(1,1),(2p,3)}}{C^{+-}_{(2,1),(2p-1,3)}} &= \frac{(3+2p)(8a^2+b^2(4p^2-1))}{(5+2p)(8a^2+b^2(2p-3)(2p+1))} \\ &\neq \frac{(2p-3)(8a^2+b^2(4p^2-1))}{(2p-7)(8a^2+b^2(2p-3)(2p+1))} = \frac{C^{++}_{(1,1),(2p,3)}}{C^{++}_{(2,1)(2p-1,3)}};\end{aligned}$$

$$\begin{aligned}\frac{C^{-+}_{(1,1),(2,2p)}}{C^{-+}_{(1,2)(2,2p-1)}} &= \frac{2(1+p)(3b^2+a^2(4p^2-1))}{(2p+3)(3b^2+a^2(2p-3)(2p+1))} \\ &\neq \frac{2(p-1)(3b^2+a^2(4p^2-1))}{(2p-5)(3b^2+a^2(2p-3)(2p+1))} = \frac{C^{++}_{(1,1),(2,2p)}}{C^{++}_{(1,2)(2,2p-1)}};\end{aligned}$$

and

$$\begin{aligned}\frac{C^{+-}_{(1,1),(2p,2)}}{C^{+-}_{(2,1),(2p-1,2)}} &= \frac{2(1+p)(3a^2+b^2(4p^2-1))}{(3+2p)(3a^2+b^2(2p-3)(2p+1))} \\ &\neq \frac{2(p-1)(3a^2+b^2(4p^2-1))}{(-5+2p)(3a^2+b^2(2p-3)(2p+1))} = \frac{C^{++}_{(1,1),(2p,2)}}{C^{++}_{(2,1)(2p-1,2)}}.\end{aligned}$$

Then the $(2p+1)^2-1$ vectors in $F_{S_{j+1}} \cup \{e_k \mid k \in \mathcal{K}^{j+1}\}$ are linearly independent and span

$$span\{W_k \mid 1 \leq k_1, k_2 \leq 2p+1\} \setminus \{(2p+1, 2p+1)\} = span(g^{j+1}).$$

Therefore $span(g^{j+1}) \subseteq L^{j+2}$.

COMPUTATIONS FOR THE CASE OF RECTANGLE (using MATHEMATICA 5.2)

--- The wedge and vee maps ---

```
w[m_, n_] := m[[1]] n[[2]] - n[[1]] m[[2]]
v[m_, n_] := m[[1]] n[[2]] + n[[1]] m[[2]]

w[{p, q}, {r, s}]
v[{p, q}, {r, s}]
```

$-q\,r + p\,s$

$q\,r + p\,s$

--- The terms $(n\,\alpha\beta\,m)^{+}$ **---**

```
MM[m_, n_] := {Abs[n[[1]] - m[[1]]], Abs[n[[2]] - m[[2]]]}
MP[m_, n_] := {Abs[n[[1]] - m[[1]]], Abs[n[[2]] + m[[2]]]}
PM[m_, n_] := {Abs[n[[1]] + m[[1]]], Abs[n[[2]] - m[[2]]]}
PP[m_, n_] := {Abs[n[[1]] + m[[1]]], Abs[n[[2]] + m[[2]]]}


MM[{p, q}, {r, s}]
MP[{p, q}, {r, s}]
PM[{p, q}, {r, s}]
PP[{p, q}, {r, s}]
```

{Abs[-p + r], Abs[-q + s]}

{Abs[-p + r], Abs[q + s]}

{Abs[p + r], Abs[-q + s]}

{Abs[p + r], Abs[q + s]}

--- The eigenvalues ---

$$\text{Bar[m\_]} := \pi\text{\^{}2}\left(\frac{\text{m[[1]]\^{}2}}{\text{a\^{}2}} + \frac{\text{m[[2]]\^{}2}}{\text{b\^{}2}}\right)$$

```
Bar[{p, q}]
```

$$\pi^2\left(\frac{p^2}{a^2} + \frac{q^2}{b^2}\right)$$

--- The coefficients C of -Bu ---

$$\text{CMM[m\_, n\_]} := \frac{\pi\text{\^{}2}}{4\,a\,b}\,\frac{\text{w[m, n]}}{\text{Bar[MM[m, n]]}}\,(\text{Bar[n]} - \text{Bar[m]})\,(\text{Sign[n[[2]] - m[[2]]]})\,(\text{Sign[n[[1]] - m[[1]]]})$$

$$\text{CMP[m\_, n\_]} := -\frac{\pi\text{\^{}2}}{4\,a\,b}\,\frac{\text{v[m, n]}}{\text{Bar[MP[m, n]]}}\,(\text{Bar[n]} - \text{Bar[m]})\,(\text{Sign[n[[1]] - m[[1]]]})$$

$$\text{CPM[m\_, n\_]} := \frac{\pi\text{\^{}2}}{4\,a\,b}\,\frac{\text{v[m, n]}}{\text{Bar[PM[m, n]]}}\,(\text{Bar[n]} - \text{Bar[m]})\,(\text{Sign[n[[2]] - m[[2]]]})$$

$$\text{CPP[m\_, n\_]} := -\frac{\pi\text{\^{}2}}{4\,a\,b}\,\frac{\text{w[m, n]}}{\text{Bar[PP[m, n]]}}\,(\text{Bar[n]} - \text{Bar[m]})$$

```
FullSimplify[CMM[{p, q}, {r, s}]]
FullSimplify[CMP[{p, q}, {r, s}]]
FullSimplify[CPM[{p, q}, {r, s}]]
FullSimplify[CPP[{p, q}, {r, s}]]
```

$$\frac{\pi^2 (q r - p s) (b^2 (p - r) (p + r) + a^2 (q - s) (q + s)) \text{Sign}[-p + r] \text{Sign}[-q + s]}{4 (a b^3 \text{Abs}[p - r]^2 + a^3 b \text{Abs}[q - s]^2)}$$

$$\frac{\pi^2 (q r + p s) (b^2 (p - r) (p + r) + a^2 (q - s) (q + s)) \text{Sign}[-p + r]}{4 (a b^3 \text{Abs}[p - r]^2 + a^3 b \text{Abs}[q + s]^2)}$$

$$-\frac{\pi^2 (q r + p s) (b^2 (p - r) (p + r) + a^2 (q - s) (q + s)) \text{Sign}[-q + s]}{4 (a b^3 \text{Abs}[p + r]^2 + a^3 b \text{Abs}[q - s]^2)}$$

$$-\frac{\pi^2 (q r - p s) (b^2 (p - r) (p + r) + a^2 (q - s) (q + s))}{4 (a b^3 \text{Abs}[p + r]^2 + a^3 b \text{Abs}[q + s]^2)}$$

--- The square ---

```
Simplify[Det[{
   {CMP[{1, 1}, {2, 4}], CPM[{1, 1}, {2, 4}], CPP[{1, 1}, {2, 4}]},
   {CMP[{1, 2}, {2, 3}], 0, CPP[{1, 2}, {2, 3}]},
   {CMP[{1, 4}, {2, 1}], CPM[{1, 4}, {2, 1}], CPP[{1, 4}, {2, 1}]}
  }]]
```

$$-\frac{15 (125 a^6 + 75 a^4 b^2 - 5 a^2 b^4 - 3 b^6) \pi^6}{16 a^3 b^3 (625 a^6 + 875 a^4 b^2 + 259 a^2 b^4 + 9 b^6)}$$

------------- The inductive steps: ------------------------

$$\textbf{FullSimplify}\left[\frac{\textbf{CMP}[\{1, 1\}, \{2, 2p+1\}]}{\textbf{CMP}[\{1, 3\}, \{2, 2p-1\}]}\right]$$

$$\textbf{FullSimplify}\left[\frac{\textbf{CPP}[\{1, 1\}, \{2, 2p+1\}]}{\textbf{CPP}[\{1, 3\}, \{2, 2p-1\}]}\right]$$

$$\frac{(3 + 2 p) (3 b^2 + 4 a^2 p (1 + p))}{(5 + 2 p) (3 b^2 + 4 a^2 (-2 + p) (1 + p))}$$

$$\frac{(-1 + 2 p) (3 b^2 + 4 a^2 p (1 + p))}{(-7 + 2 p) (3 b^2 + 4 a^2 (-2 + p) (1 + p))}$$

$$\textbf{FullSimplify}\left[\frac{\textbf{CPM}[\{1, 1\}, \{2p+1, 2\}]}{\textbf{CPM}[\{3, 1\}, \{2p-1, 2\}]}\right]$$

$$\textbf{FullSimplify}\left[\frac{\textbf{CPP}[\{1, 1\}, \{2p+1, 2\}]}{\textbf{CPP}[\{3, 1\}, \{2p-1, 2\}]}\right]$$

$$\frac{(3 + 2 p) (3 a^2 + 4 b^2 p (1 + p))}{(5 + 2 p) (3 a^2 + 4 b^2 (-2 + p) (1 + p))}$$

$$\frac{(-1 + 2 p) (3 a^2 + 4 b^2 p (1 + p))}{(-7 + 2 p) (3 a^2 + 4 b^2 (-2 + p) (1 + p))}$$

for j even; and ---

```
FullSimplify[CMP[{1, 1}, {3, 2p}] / CMP[{1, 2}, {3, 2p - 1}]]
FullSimplify[CPP[{1, 1}, {3, 2p}] / CPP[{1, 2}, {3, 2p - 1}]]
```

$$\frac{(3+2\,p)\,(8\,b^2+a^2\,(-1+4\,p^2))}{(5+2\,p)\,(8\,b^2+a^2\,(-3+2\,p)\,(1+2\,p))}$$

$$\frac{(-3+2\,p)\,(8\,b^2+a^2\,(-1+4\,p^2))}{(-7+2\,p)\,(8\,b^2+a^2\,(-3+2\,p)\,(1+2\,p))}$$

```
FullSimplify[CPM[{1, 1}, {2p, 3}] / CPM[{2, 1}, {2p - 1, 3}]]
FullSimplify[CPP[{1, 1}, {2p, 3}] / CPP[{2, 1}, {2p - 1, 3}]]
```

$$\frac{(3+2\,p)\,(8\,a^2+b^2\,(-1+4\,p^2))}{(5+2\,p)\,(8\,a^2+b^2\,(-3+2\,p)\,(1+2\,p))}$$

$$\frac{(-3+2\,p)\,(8\,a^2+b^2\,(-1+4\,p^2))}{(-7+2\,p)\,(8\,a^2+b^2\,(-3+2\,p)\,(1+2\,p))}$$

```
FullSimplify[CMP[{1, 1}, {2, 2p}] / CMP[{1, 2}, {2, 2p - 1}]]
FullSimplify[CPP[{1, 1}, {2, 2p}] / CPP[{1, 2}, {2, 2p - 1}]]
```

$$\frac{2\,(1+p)\,(3\,b^2+a^2\,(-1+4\,p^2))}{(3+2\,p)\,(3\,b^2+a^2\,(-3+2\,p)\,(1+2\,p))}$$

$$\frac{2\,(-1+p)\,(3\,b^2+a^2\,(-1+4\,p^2))}{(-5+2\,p)\,(3\,b^2+a^2\,(-3+2\,p)\,(1+2\,p))}$$

```
FullSimplify[CPM[{1, 1}, {2p, 2}] / CPM[{2, 1}, {2p - 1, 2}]]
FullSimplify[CPP[{1, 1}, {2p, 2}] / CPP[{2, 1}, {2p - 1, 2}]]
```

$$\frac{2\,(1+p)\,(3\,a^2+b^2\,(-1+4\,p^2))}{(3+2\,p)\,(3\,a^2+b^2\,(-3+2\,p)\,(1+2\,p))}$$

$$\frac{2\,(-1+p)\,(3\,a^2+b^2\,(-1+4\,p^2))}{(-5+2\,p)\,(3\,a^2+b^2\,(-3+2\,p)\,(1+2\,p))}$$

--- for j odd.

6.4 The Hemisphere under Navier boundary conditions

Consider the "upper" Hemisphere $\mathbb{S}^2_+$ defined by:

$$\mathbb{S}^2_+ := \{(x_1,\, x_2,\, x_3) \in \mathbb{S}^2 \mid x_3 > 0\}.$$

Put $r := \sqrt{x_1^2 + x_2^2 + x_3^2}$ and $\rho = r^{-1}$.

Lemma 6.4.1. *All the polynomials $P = (\partial_1^p \partial_2^q \partial_3^s \rho) r^{2n+1}$, with $p+q+s = n$, are sums of monomials of the form $C x_1^{p_1} x_2^{q_1} x_3^{s_1}$ where $C \in \mathbb{Z}$ is an integer; $p_1,\, q_1,\, s_1 \in \mathbb{N}$ are natural and we have the relations $p \equiv p_1 (\mod 2)$, $q \equiv q_1 (\mod 2)$ and $s \equiv s_1 (\mod 2)$.*

Proof. We prove the lemma by induction on n.

For $n = 0$ we have only one monomial: the constant 1.

Suppose that the statement is true for some $n-1 \geq 0$. Let $P = (\partial_1^p \partial_2^q \partial_3^s \rho) r^{2n+1}$, with $n = p+q+s$. Since $n > 0$, at least one of $p,\, q,\, s$ is positive. Suppose without loss of generality that $p > 0$. Then

$$\begin{aligned} P &= (\partial_1^p \partial_2^q \partial_3^s \rho) r^{2(n-1)+1} r^2 \\ &= \left[\partial_1 \left((\partial_1^{p-1} \partial_2^q \partial_3^s \rho) r^{2(n-1)+1} \right)\right] r^2 - \left[(\partial_1^{p-1} \partial_2^q \partial_3^s \rho) \partial_1 (r^{2(n-1)+1}) \right] r^2. \end{aligned}$$

For the last term we find

$$\begin{aligned} -\left[(\partial_1^{p-1} \partial_2^q \partial_3^s \rho) \partial_1 (r^{2(n-1)+1}) \right] r^2 &= - \left[(2(n-1)+1) \partial_1^{p-1} \partial_2^q \partial_3^s \rho \right] r^{2(n-1)} (x_1 \rho) r^2 \\ &= - \left[(2(n-1)+1) \partial_1^{p-1} \partial_2^q \partial_3^s \rho \right] r^{2(n-1)+1} x_1. \end{aligned}$$

By induction hypothesis $Q = (\partial_1^{p-1} \partial_2^q \partial_3^s \rho) r^{2(n-1)+1}$ is a sum of monomials of the form $D x_1^{p_1} x_2^{q_1} q x_3^{s_1}$ with D integer, $p_1,\, q_1,\, s_1$ natural and $p-1 \equiv p_1 (\mod 2)$, $q \equiv q_1 (\mod 2)$ and $s \equiv s_1 (\mod 2)$. By a simple observation we see that P is necessarily a sum of monomials of the form $C x_1^{p_1} x_2^{q_1} x_3^{s_1}$ with C integer, $p \equiv p_1 (\mod 2)$, $q \equiv q_1 (\mod 2)$ and $s \equiv s_1 (\mod 2)$. ☐

The spherical harmonics that are restrictions of the polynomials $P = (\partial_1^p \partial_2^q \partial_3^s \rho) r^{2n+1}$, with $n = p+q+s$, form a complete system for $L^2(\mathbb{S}^2)$. From that, we have that the restrictions of the polynomials $P = (\partial_1^p \partial_2^q \partial_3^s \rho) r^{2n+1}$ with s odd form a complete basis for $L^2(\mathbb{S}^2_+)$. The elements of that basis vanish on the boundary $\mathbb{S}^2_0 := \partial \mathbb{S}^2_+ = \{(x_1,\, x_2,\, x_3) \in \mathbb{S}^2 \mid x_3 = 0\}$. Note that given a scalar function defined in $\mathbb{S}^2_+$ and vanishing on $\mathbb{S}^2_0$, we extend it to all $\mathbb{S}^2$ by defining $f(x_1,\, x_2,\, -x_3) := -f(x_1,\, x_2,\, x_3)$ for all $(x_1,\, x_2,\, x_3) \in \mathbb{S}^2$. For a polynomial P whose monomials are of the form $C x_1^p x_2^q x_3^s$ with s even we have $(f,\, P)_{L^2(\mathbb{S}^2)} = \int_{\mathbb{S}^2} f P \, d\mathbb{S}^2 = 0$. Therefore on the Sphere, $f = \sum_{i=1}^{+\infty} C_i (f,\, P_i)_{L^2(\mathbb{S}^2)} P_i$ for polynomials P_i whose monomials $C x_1^p x_2^q x_3^s$ satisfy: s is odd.

For odd $s \geq 3$, we may rewrite any monomial $C x_1^p x_2^q x_3^s$ in the Sphere as $C(x_1^p x_2^q - x_1^{p+2} x_2^q - x_1^p x_2^{q+2}) x_3^{s-2}$ so; any such monomial may be rewritten as a sum $C x_1^p x_2^q x_3^s = \sum_{i=1}^m P_i(x_1,\, x_2) x_3$ where $P_i(x_1,\, x_2)$ are polynomials of degree less or equal to $p+q+s-1$. Therefore we have the following proposition:

Proposition 6.4.2. *The system $\{Px_3\}$ formed by functions vanishing on the boundary $\mathbb{S}^2_0$, where $P = P(x_1, x_2)$ runs over all homogeneous polynomials in the variables x_1 and x_2, is complete in $L^2(\mathbb{S}^2_+)$.*

The metric tensor in $\mathbb{S}^2_+$ inherited from the Euclidean metric tensor in $\mathbb{R}^3$ is

$$\begin{aligned} g_{ij}dx^i \otimes dx^j =& \frac{1-x_2^2}{x_3^2}dx^1 \otimes dx^1 + \frac{x_1x_2}{x_3^2}dx^1 \otimes dx^2 \\ &+ \frac{x_1x_2}{x_3^2}dx^2 \otimes dx^1 + \frac{1-x_1^2}{x_3^2}dx^2 \otimes dx^2. \end{aligned}$$

Remark 6.4.1. *We are considering the chart*

$$(x^1,\, x^2) \mapsto (x^1,\, x^2,\, \sqrt{1-(x^1)^2-(x^2)^2})$$

and we identify x_1 with x^1. We write dx^i instead of dx_i just to preserve some notation from Riemannian geometry.

The inverse of the matrix $[g_{ij}] = \frac{1}{x_3^2}\begin{bmatrix} 1-x_2^2 & x_1x_2 \\ x_1x_2 & 1-x_1^2 \end{bmatrix}$ is given by the matrix $[g^{ij}] = \begin{bmatrix} 1-x_1^2 & -x_1x_2 \\ -x_1x_2 & 1-x_2^2 \end{bmatrix}$.

We recall that the Laplace-de Rham operator[5] (or simply Laplacean) Δ applied to a scalar function f defined in $\mathbb{S}^2_+$ gives

$$\Delta f = -x_3\partial_i\left(\frac{1}{x_3}g^{ij}\partial_j f\right).$$

We will need to know the evaluation of the Laplacean on monomials of the form $x_1^px_2^qx_3$; we start with a lemma:

Lemma 6.4.3. *For $p, q \geq 2$ we have*

$$\begin{aligned} \Delta x_1^p &= p(p+1)x_1^p - (p-1)px_1^{p-2}; \\ \Delta x_2^q &= q(q+1)x_2^q - (q-1)qx_2^{q-2}; \\ \Delta x_1^px_2 &= (p+1)(p+2)x_1^px_2 - (p-1)px_1^{p-2}x_2; \\ \Delta x_1x_2^q &= (q+1)(q+2)x_1x_2^q - (q-1)qx_1x_2^{q-2}; \\ \Delta x_1^px_2^q &= (p+q)(p+q+1)x_1^px_2^q - (p-1)px_1^{p-2}x_2^q - (q-1)qx_1^px_2^{q-2}. \end{aligned}$$

[5] In general, for functions, defined as $- * d * df$; where $*$ is the Hodge map and d the differential map.

Proof. By direct computation

$$\begin{aligned}
\Delta x_1^p &= -x_3\partial_i\left(\frac{1}{x_3}g^{ij}\partial_j x_1^p\right) = -x_3\partial_i\left(\frac{1}{x_3}g^{i1}px_1^{p-1}\right)\\
&= -x_3\left[\partial_1\left(\frac{1}{x_3}(1-x_1^2)px_1^{p-1}\right)+\partial_2\left(\frac{1}{x_3}(-x_1x_2)px_1^{p-1}\right)\right]\\
&= -x_3\left[\frac{x_1}{x_3^3}(px_1^{p-1}-px_1^{p+1})+\frac{1}{x_3}\left((p-1)px_1^{p-2}-p(p+1)x_1^p\right)\right]\\
&\quad - x_3\left[\frac{x_2}{x_3^3}(-px_1^px_2)+\frac{1}{x_3}(-px_1^p)\right]\\
&= p(p+2)x_1^p-(p-1)px_1^{p-2}-\frac{1}{x_3^2}p(-x_1^{p+2}+x_1^p-x_1^px_2^2)\\
&= p(p+2)x_1^p-(p-1)px_1^{p-2}-px_1^p = p(p+1)x_1^p-(p-1)px_1^{p-2}
\end{aligned}$$

Similarly we can prove that

$$\Delta x_2^q = q(q+1)x_2^q-(q-1)qx_2^{q-2}.$$

To compute $\Delta x_1^p x_2$ first we observe that

$$\begin{aligned}
\Delta fg &= -x_3\partial_i\left(\frac{1}{x_3}g^{ij}((\partial_j f)g+f(\partial_j g))\right)\\
&= (\Delta f)g - g^{ij}(\partial_j f)(\partial_i g) - g^{ij}(\partial_i f)(\partial_j g)+f(\Delta g)\\
&= (\Delta f)g - 2g^{ij}(\partial_j f)(\partial_i g)+f(\Delta g).
\end{aligned}$$

Therefore

$$\begin{aligned}
\Delta x_1^p x_2 =&(\Delta x_1^p)x_2 - 2g^{12}px_1^{p-1}+2x_1^px_2\\
=&[p(p+1)+2p+2]x_1^px_2-(p-1)px_1^{p-2}x_2\\
=&(p+1)(p+2)x_1^px_2-(p-1)px_1^{p-2}x_2.
\end{aligned}$$

and, similarly we can prove that

$$\Delta x_1x_2^q = (q+1)(q+2)x_1x_2^q-(q-1)qx_1x_2^{q-2}.$$

Finally

$$\begin{aligned}
&\Delta x_1^px_2^q = (\Delta x_1^p)x_2^q - 2g^{12}px_1^{p-1}qx_2^{q-1}+x_1^p(\Delta x_2^p)\\
=&[p(p+1)+2pq+q(q+1)]x_1^px_2^q-(p-1)px_1^{p-2}x_2^q-(q-1)qx_1^px_2^{q-2}\\
=&(p+q)(p+q+1)x_1^px_2^p-(p-1)px_1^{p-2}x_2^q-(q-1)qx_1^px_2^{q-2}.
\end{aligned}$$

□

Proposition 6.4.4. *Let p, $q \geq 2$ be two natural numbers. The Laplacean takes the following values, in the monomials forming the complete system:*

$$\Delta x_3 = 2x_3; \quad \Delta x_1x_3 = 6x_1x_3; \quad \Delta x_2x_3 = 6x_2x_3; \quad \Delta x_1x_2x_3 = 12x_1x_2x_3;$$
$$\Delta x_1^p x_3 = \Big((p+1)(p+2)x_1^p - p(p+1)x_1^{p-2}\Big)\, x_3;$$
$$\Delta x_2^q x_3 = \Big((q+1)(q+2)x_2^q - q(q+1)x_2^{q-2}\Big)\, x_3;$$
$$\Delta x_1^p x_2 x_3 = \Big((p+2)(p+3)x_1^p x_2 - p(p+1)x_1^{p-2}x_2\Big)\, x_3;$$
$$\Delta x_1 x_2^q x_3 = \Big((q+2)(q+3)x_1 x_2^q - q(q+1)x_1 x_2^{q-2}\Big)\, x_3;$$
$$\Delta x_1^p x_2^q x_3 = \Big((p+q+1)(p+q+2)x_1^p x_2^q - p(p-1)x_1^{p-2}x_2^q - q(q-1)x_1^p x_2^{q-2}\Big)\, x_3.$$

Proof. The first identities follow from the fact that x_3, x_1x_3, x_2x_3 and $x_1x_2x_3$ are spherical harmonics of degree 1, 2, 2 and 3 respectively. Recall that restrictions to the Sphere of harmonic and homogeneous polynomials in $\mathbb{R}^3$ are eigenfunctions of the spherical Laplacean associated with the eigenvalue $k(k+1)$, where k is the degree of the polynomial.

The remaining identities can be obtained by the formula

$$\Delta f x_3 = (\Delta f)x_3 - 2g^{ij}(\partial_i f)(\partial_j x_3) + 2f x_3$$

and by the expressions in lemma 6.4.3; we present only the computations for the last case:

$$\begin{aligned}
\Delta x_1^p x_2^q x_3 =& (\Delta x_1^p x_2^q)x_3 - 2(1-x_1^2)p x_1^{p-1}x_2^q \frac{-x_1}{x_3} + 2(x_1x_2)p x_1^{p-1}x_2^q\frac{-x_2}{x_3} \\
&+ 2(x_1x_2)q x_1^p x_2^{q-1}\frac{-x_1}{x_3} - 2(1-x_2^2)q x_1^p x_2^{q-1}\frac{-x_2}{x_3} + 2x_1^p x_2^q x_3 \\
=& (\Delta x_1^p x_2^q + 2x_1^p x_2^q)x_3 - \frac{1}{x_3}2p x_1^p x_2^q(-1+x_1^2+x_2^2) - \frac{1}{x_3}2q x_1^p x_2^q(x_1^2-1+x_2^2) \\
=& \bigg(\Big((p+q)(p+q+1)+2+2p+2q\Big)x_1^p x_2^q - (p-1)p x_1^{p-2}x_2^q - (q-1)q x_1^p x_2^{q-2}\bigg)x_3 \\
=& \Big((p+q+1)(p+q+2)x_1^p x_2^q - (p-1)p x_1^{p-2}x_2^q - (q-1)q x_1^p x_2^{q-2}\Big)x_3.
\end{aligned}$$

□

As we may deduce from proposition 6.4.4 the Laplacean of a polynomial of the form $P(x_1, x_2)x_3$, where $P(x_1, x_2)$ is a polynomial in the variables x_1 and x_2, is a polynomial $Q(x_1, x_2)x_3$ where Q has the same degree as P. We also have

Proposition 6.4.5. *The Laplacean is surjective in the space of the polynomials P_m, formed by polynomials $P(x_1, x_2)x_3$, with $P(x_1, x_2)$ a polynomial of fixed degree m. Moreover, Δ is invertible and*

$$(\Delta)^{-1}x_1^p x_2^q x_3 = \frac{1}{(p+q+1)(p+q+2)}x_1^p x_2^q x_3 + R(x_1, x_2)x_3$$

where the remainder $R(x_1, x_2)$ is a polynomial of degree less or equal than $p+q-2$.

Proof. The invertibility of $\Delta : P_m \to \Delta(P_m)$ follows from the injectivity of Δ; recall that the system $\begin{cases} \Delta f = 0 & \text{in } \mathbb{S}^2_+ \\ f = 0 & \text{on the boundary } \mathbb{S}^2_0 \end{cases}$ has only the trivial solution $f = 0$. [6]

We prove the surjectiveness by induction: the surjectiveness clearly holds for $m = 0, 1$; suppose now that the surjectiveness holds for $m - 1 \geq 0$ and fix a monomial $x_1^p x_2^q x_3$ with $p + q = m$. Then $\Delta \frac{1}{(p+q+1)(p+q+2)} x_1^p x_2^q x_3 = x_1^p x_2^q x_3 + \bar{R}(x_1, x_2)x_3$ where $\bar{R}(x_1, x_2)x_3$ is a polynomial of degree $m - 2$ so, by induction hypothesis there exists $R(x_1, x_2)x_3$ of degree $m - 2$ such that $\Delta R(x_1, x_2)x_3 = -\bar{R}(x_1, x_2)x_3$. Therefore

$$\Delta \left[\frac{1}{(p+q+1)(p+q+2)} x_1^p x_2^q x_3 + R(x_1, x_2)x_3 \right] = x_1^p x_2^q x_3$$

and R has degree less or equal to $m - 2$. $\square$

We recall that the Poisson bracket $\{f, g\}$, between two scalar functions defined on $\mathbb{S}^2_+$, may be computed as

$$\{f, g\} = \langle x, \nabla_x f, \nabla_x g \rangle$$

where $x = (x_1, x_2, x_3) \in \mathbb{S}^2_+$, ∇_x is the gradient operator in $\mathbb{R}^3$ and $\langle a, b, c \rangle$ is the determinant of the matrix whose columns are a, b, and c. [7]

Lemma 6.4.6. *Consider the eigenfunctions x_3 and x_1x_3 of Δ and consider a monomial $x_1^p x_2^q x_3$. We have the following identities for $p, q \geq 1$*

$$\begin{aligned}
\{x_1^p x_2^q x_3,\, x_3\} &= \left(q x_1^{p+1} x_2^{q-1} - p x_1^{p-1} x_2^{q+1} \right) x_3; \\
\{x_1^p x_3,\, x_3\} &= -p x_1^{p-1} x_2 x_3; \\
\{x_2^q x_3,\, x_3\} &= q x_1 x_2^{q-1} x_3; \\
\{x_1^p x_2^q x_3,\, x_1 x_3\} &= \left(2q x_1^{p+2} x_2^{q-1} + (q - p + 1) x_1^p x_2^{q+1} - q x_1^p x_2^{q-1} \right) x_3; \\
\{x_1^p x_3,\, x_1 x_3\} &= (1 - p) x_1^p x_2 x_3; \\
\{x_2^q x_3,\, x_1 x_3\} &= \left(2q x_1^2 x_2^{q-1} + (q+1) x_2^{q+1} - q x_2^{q-1} \right) x_3.
\end{aligned}$$

Proof. The proof follows by direct computation; we present the computations for the first and fourth cases, the other cases are completely analogous.

$$\{x_1^p x_2^q x_3,\, x_3\} = \det \begin{bmatrix} x_1 & x_2 & x_3 \\ p x_1^{p-1} x_2^q x_3 & q x_1^p x_2^{q-1} x_3 & x_1^p x_2^q \\ 0 & 0 & 1 \end{bmatrix} = \left(q x_1^{p+1} x_2^{q-1} - p x_1^{p-1} x_2^{q+1} \right) x_3;$$

[6] Given a function f vanishing on the boundary Γ of a manifold Ω with $\Delta f = 0$ we find that $0 = (f, \Delta f)_{L^2(\Omega)} = -\int_\Omega f * d * df \, d\Omega = -\int_\Omega f d * df = -\int_\Omega d(f * df) - \int_\Omega df \wedge *df = -\int_\Gamma f * df - \int_\Omega i_{\nabla f} df \, d\Omega = -\int_\Gamma f * df + \int_\Omega g(df, df) \, d\Omega$. The integral $\int_\Gamma f * df$ vanish so, the differential df must vanish which implies that f is constant and so, necessarily $f = 0$ in Ω, because $f = 0$ on Γ.

[7] Clearly we are supposing that f, g are restrictions to the Sphere of some $\tilde{f}$, $\tilde{g}$ defined in some neighborhood of $\mathbb{S}^2_+$ in $\mathbb{R}^3$. To be more precise we should write $\nabla_x \tilde{f}$ and $\nabla_x \tilde{g}$ in the places of $\nabla_x f$ and $\nabla_x g$.

$$
\begin{aligned}
\{x_1^p x_2^q x_3,\, x_1 x_3\} =& \det \begin{bmatrix} x_1 & x_2 & x_3 \\ p x_1^{p-1} x_2^q x_3 & q x_1^p x_2^{q-1} x_3 & x_1^p x_2^q \\ x_3 & 0 & x_1 \end{bmatrix} \\
=& x_1 \left(q x_1^{p+1} x_2^{q-1} - p x_1^{p-1} x_2^{q+1} \right) x_3 + x_3 \left(x_1^p x_2^{q+1} - q x_1^p x_2^{q-1} x_3^2 \right) \\
=& \left(q x_1^{p+2} x_2^{q-1} + (1-p) x_1^p x_2^{q+1} - q x_1^p x_2^{q-1} + q x_1^{p+2} x_2^q + q x_1^p x_2^{q+1} \right) x_3 \\
=& \left(2q x_1^{p+2} x_2^{q-1} + (q-p+1) x_1^p x_2^{q+1} - q x_1^p x_2^{q-1} \right) x_3.
\end{aligned}
$$

□

Corollary 6.4.7. *Again for $p,\, q \geq 1$; "projecting", by P_d, the last expressions on the space of the polynomials $M_d := span\{x_1^{p_1} x_2^{q_1} x_3 \mid p_1 + q_1 = d\}$, where d is the maximum degree of the expression to be projected, i.e., eliminating low order monomials, we have*

$$
\begin{aligned}
P_{p+q}\{x_1^p x_2^q x_3,\, x_3\} &= \left(q x_1^{p+1} x_2^{q-1} - p x_1^{p-1} x_2^{q+1} \right) x_3; \\
P_p\{x_1^p x_3,\, x_3\} &= -p x_1^{p-1} x_2 x_3; \\
P_q\{x_2^q x_3,\, x_3\} &= q x_1 x_2^{q-1} x_3; \\
P_{p+q+1}\{x_1^p x_2^q x_3,\, x_1 x_3\} &= \left(2q x_1^{p+2} x_2^{q-1} + (q-p+1) x_1^p x_2^{q+1} \right) x_3; \\
P_{p+1}\{x_1^p x_3,\, x_1 x_3\} &= (1-p) x_1^p x_2 x_3; \\
P_{q+1}\{x_2^q x_3,\, x_1 x_3\} &= \left(2q x_1^2 x_2^{q-1} + (q+1) x_2^{q+1} \right) x_3.
\end{aligned}
$$

In the recursive step of the definition of l-saturating set, the new vorticities appear from the computation of sums of brackets

$$
\{\Delta^{-1} v_i,\, v\} + \{\Delta^{-1} v,\, v_i\}. \tag{6.12}
$$

For the monomial $x_1^p x_2^q x_3$ with $p + q \geq 1$, using proposition 6.4.5, we obtain

$$
\begin{aligned}
&\{\Delta^{-1} x_3,\, x_1^p x_2^q x_3\} + \{\Delta^{-1} x_1^p x_2^q x_3,\, x_3\} \\
&= \left(-\frac{1}{2} + \frac{1}{(p+q+1)(p+q+2)} \right) \{x_1^p x_2^q x_3,\, x_3\} - \{R_1(x_1,\, x_2) x_3,\, x_3\}
\end{aligned}
$$

and

$$
\begin{aligned}
&\{\Delta^{-1} x_1 x_3,\, x_1^p x_2^q x_3\} + \{\Delta^{-1} x_1^p x_2^q x_3,\, x_1 x_3\} \\
&= \left(-\frac{1}{6} + \frac{1}{(p+q+1)(p+q+2)} \right) \{x_1^p x_2^q x_3,\, x_1 x_3\} - \{R_2(x_1,\, x_2) x_3,\, x_1 x_3\}
\end{aligned}
$$

where R_1 and R_2 are polynomials of degree less or equal to $p+q-2$. Thus $\{R_1(x_1,\, x_2)x_3,\, x_3\}$ and $\{R_2(x_1,\, x_2)x_3,\, x_1x_3\}$ are polynomials of degree less or equal to $1 + [(p+q-2+1)-1] + [1-1] = p+q-1$ and $1 + [(p+q-2+1)-1] + [2-1] = p+q$ respectively.

Therefore, eliminating low order monomials, we obtain

Proposition 6.4.8. *For a monomial $x_1^p x_2^q x_3$:*

$$\begin{aligned} P_{p+q}\left(\{\Delta^{-1}x_3,\, x_1^p x_2^q x_3\} + \{\Delta^{-1}x_1^p x_2^q x_3,\, x_3\}\right) =& C_1\{x_1^p x_2^q x_3,\, x_3\}, \\ P_{p+q+1}\left(\{\Delta^{-1}x_1 x_3,\, x_1^p x_2^q x_3\} + \{\Delta^{-1}x_1^p x_2^q x_3,\, x_3\}\right) =& C_2\{x_1^p x_2^q x_3,\, x_1 x_3\}, \end{aligned}$$

where $C_1 = \left(-\frac{1}{2} + \frac{1}{(p+q+1)(p+q+2)}\right)$ *and* $C_2 = \left(-\frac{1}{6} + \frac{1}{(p+q+1)(p+q+2)}\right)$*. Note that C_1 is nonzero for $p+q \geq 1$ and, C_2 is nonzero for $p+q \geq 2$.*

Now we give the saturating set:

Theorem 6.4.9. *The set of eigenfunctions $h = \{x_3,\, x_1x_3,\, (5x_1^2-1)x_3\}$ is a $l^\perp$-saturating set of vorticities.* [8]

Proof. First of all, we note that proposition 6.4.8 somehow reduces the problem of looking for a saturating set, to the computation of some single brackets instead of the computation of sums of brackets like (6.12).

Let $\left(L^{\perp,n}\right)_{n\in\mathbb{N}}$ be the sequence given by the definition of $l^\perp$-saturating set.

First step: *The monomial x_3 is in $L^{\perp,0}$.* Trivial.

Second step: *The monomials x_1x_3 and x_2x_3 are in $L^{\perp,1}$.* By

$$\{x_1x_3,\, x_3\} = -x_2x_3.$$

Third step: *The monomials $x_1^p x_2^q x_3$ with $p+q=2$ are in $L^{\perp,2}$.* By

$$\begin{aligned} \{(5x_1^2-1)x_3,\, x_3\} &= 5\{x_1^2x_3,\, x_3\} = -10x_1x_2x_3; \\ \{x_1x_2x_3,\, x_3\} &= (x_1^2 - x_2^2)x_3. \end{aligned}$$

Fourth step (induction): *The monomials $x_1^p x_2^q x_3$ with $p+q=m$ are in $L^{\perp,2+3(m-2)}$.* The statement is true for $m=2$; suppose it is true for given $m \geq 2$. By corollary 6.4.7 we have that, for $1 \leq i \leq m-1$,

$$\begin{aligned} P_{m+1}\{x_1^m x_3,\, x_1x_3\} &= (1-m)x_1^m x_2 x_3; \\ P_{m+1}\{x_1^{m-i}x_2^i x_3,\, x_1x_3\} &= \left(2ix_1^{m-i+2}x_2^{i-1} + (2i-m+1)x_1^{m-i}x_2^{i+1}\right)x_3; \\ P_{m+1}\{x_2^m x_3,\, x_1x_3\} &= \left(2mx_1^2x_2^{m-1} + (m+1)x_2^{m+1}\right)x_3. \end{aligned}$$

Consider two cases "m is even" and "m is odd".

Case m is even:

Step 1: Consider the family of functions

$$\left\{P_{m+1}\{x_1^{m-2j}x_2^{2j}x_3,\, x_1x_3\} \mid j = 0,\, 1,\, \ldots,\, \frac{m}{2}\right\}. \tag{6.13}$$

For $j=0$ we have $P_{m+1}\{x_1^m x_3,\, x_1x_3\} = (1-m)x_1^m x_2 x_3$. Suppose that for $0 \leq s < \frac{m}{2} - 1$ we have that $\left\{P_{m+1}\{x_1^{m-2j}x_2^{2j}x_3,\, x_1x_3\} \mid j = 0,\, 1,\, \ldots,\, s\right\}$ span the space

$$span\{x_1^{m-2j}x_2^{2j+1}x_3 \mid j = 0,\, 1,\, \ldots,\, s\};$$

[8] Note that $(5x_1^2-1)x_3 = (5x_1^2-r^2)x_3$ on $\mathbb{S}^2_+$ and, as polynomial in $\mathbb{R}^3$, $(5x_1^2-r^2)x_3$ is homogeneous and harmonic.

then we add the function

$$\begin{aligned}&P_{m+1}\{x_1^{m-2(s+1)}x_2^{2(s+1)}x_3,\, x_1x_3\}\\&=\left(4(s+1)x_1^{m-2s}x_2^{2s+1}+(4(s+1)-m+1)x_1^{m-2(s+1)}x_2^{2s+3}\right)x_3;\end{aligned}$$

note that since m is even $4(s+1)-m+1$ is nonzero.

Therefore the functions

$$\left\{P_{m+1}\{x_1^{m-2j}x_2^{2j}x_3,\, x_1x_3\} \mid j=0,\,1,\,\ldots,\,s+1\right\}$$

are in $L^{\perp,2+3(m-2)+1}$ and span the space

$$span\{x_1^{m-2j}x_2^{2j+1}x_3 \mid j=0,\,1,\,\ldots,\,s+1\}.$$

By Induction we may replace $s+1$ by $\frac{m}{2}-1$.

Finally we add the function

$$P_{m+1}\{x_2^m x_3,\, x_1x_3\}=\left(2mx_1^2x_2^{m-1}+(m+1)x_2^{m+1}\right)x_3,$$

and conclude that

$$span\left\{x_1^{m-2j}x_2^{2j+1}x_3 \mid j=0,\,\ldots,\,\frac{m}{2}\right\}\subseteq L^{2+3(m-2)+1}.$$

Step 2: The functions

$$\left\{P_{m+1}\{x_1^{m-(2j-1)}x_2^{2j-1}x_3,\, x_1x_3\} \mid j=1,\,2,\,\ldots,\,\frac{m}{2}\right\} \tag{6.14}$$

have the expression

$$\begin{aligned}&P_{m+1}\{x_1^{m-(2j-1)}x_2^{2j-1}x_3,\, x_1x_3\}\\&=\left(2(2j-1)x_1^{m-(2j-3)}x_2^{2j-2}+(2(2j-1)-m+1)x_1^{m-(2j-1)}x_2^{2j}\right)x_3\end{aligned}$$

and belong to $L^{\perp,2+3(m-2)+1}$ and, the function

$$P_{m+1}\{x_1^m x_2x_3,\, x_3\}=\left(x_1^{m+1}-mx_1^{m-1}x_2^2\right)x_3,$$

belong to $L^{\perp,2+3(m-2)+2}$.

Now the functions

$$\begin{aligned}P_{m+1}\{x_1^m x_2x_3,\, x_3\}&=\left(x_1^{m+1}-mx_1^{m-1}x_2^2\right)x_3 \text{ and}\\P_{m+1}\{x_1^{m-1}x_2x_3,\, x_1x_3\}&=\left(2x_1^{m+1}+(1-m+1+1)x_1^{m-1}x_2^2\right)x_3\end{aligned}$$

span the space $span\{x_1^{m+1}x_3,\, x_1^{m-1}x_2^2x_3\}$ and, proceeding as before, we can conclude that the family (6.14) together with the function $\{x_1^m x_2x_3,\, x_3\}$ span the space

$$span\left\{x_1^{m-(2j-1)}x_2^{2j}x_3 \mid j=1,\,2,\,\ldots,\,\frac{m}{2}\right\}.$$

Therefore, if m is even:

$$\{x_1^p x_2^q x_3 \mid p+q=m+1\} \subseteq L^{\perp,2+3(m-2)+2} \subseteq L^{\perp,2+3((m+1)-2)}. \tag{6.15}$$

Case m is odd: We follow the proof of the previous case: only one of the coefficients $(2i-m+1)$ vanish — that one corresponding to $i_0 = \frac{m-1}{2} > 1$.

Subcase A: i_0 is even: In this case we write $i_0 = 2k$. As before all the functions

$$\begin{aligned} P_{m+1}\{x_1^m x_3,\, x_1x_3\} &= (1-m)x_1^m x_2 x_3; \\ P_{m+1}\{x_1^m x_2x_3,\, x_3\} &= \left(x_1^{m+1} - m x_1^{m-1}x_2^2\right)x_3 \text{ and} \\ P_{m+1}\{x_1^{m-1}x_2x_3,\, x_1x_3\} &= \left(2x_1^{m+1} + (1-m+1+1)x_1^{m-1}x_2^2\right)x_3 \end{aligned}$$

are in $L^{\perp,2+3(m-2)+2}$; we may "repeat" the "Step 2" of the case "m **is even**" (because that step corresponds to odd i) and we see that the family

$$\left\{P_{m+1}\{x_1^{m-(2j-1)}x_2^{2j-1}x_3,\, x_1x_3\} \mid j=1,\,2,\,\ldots,\,\frac{m+1}{2}\right\} \tag{6.16}$$

together with $P_{m+1}\{x_1^m x_2 x_3,\, x_3\} = (x_1^{m+1} - m x_1^{m-1}x_2^2)x_3$ span the space

$$span\left\{x_1^{m-(2j-1)}x_2^{2j}x_3 \mid j=0,\,1,\,\ldots,\,\frac{m+1}{2}\right\} \subseteq L^{\perp,2+3(m-2)+2}.$$

In the case of even i, we replace the "bad" function

$$\begin{aligned} P_{m+1}\{x_1^{m-2k}x_2^{2k}x_3,\, x_1x_3\} &= \left(4kx_1^{m-2(j-1)}x_2^{2k-1} + (4k-m+1)x_1^{m-2k}x_2^{2k+1}\right)x_3 \\ &= \left((m-1)x_1^{\frac{m+5}{2}}x_2^{\frac{m-3}{2}} + 0\times x_1^{\frac{m+1}{2}}x_2^{\frac{m+1}{2}}\right)x_3 \end{aligned}$$

corresponding to $i_0 = 2k = \frac{m-1}{2}$, by the function

$$\begin{aligned} P_{m+1}\{x_1^{m-(2k-1)}x_2^{2k}x_3,\, x_3\} &= \left(2kx_1^{m-2(k-1)}x_2^{2k-1} - (m-(2k-1))x_1^{m-2k}x_2^{2k+1}\right)x_3 \\ &= \frac{m-1}{2}x_1^{\frac{m+5}{2}}x_2^{\frac{m-3}{2}} - \frac{m+3}{2}x_1^{\frac{m+1}{2}}x_1^{\frac{m+1}{2}}. \end{aligned}$$

This last function is in $L^{\perp,2+3(m-2)+3}$ and, together with the family

$$\left\{P_{m+1}\{x_1^{m-2j}x_2^{2j}x_3,\, x_1x_3\} \mid j=0,\,1,\,\ldots,\,\frac{m-1}{2},\, 2j \neq \frac{m-1}{2}\right\}, \tag{6.17}$$

span the space

$$span\left\{x_1^{m-(2j)}x_2^{2j+1}x_3 \mid j=0,\,\ldots,\,\frac{m-1}{2}\right\} \subseteq L^{\perp,2+3((m+1)-2)}.$$

Therefore, if $m = 4k+1$,

$$\{x_1^p x_2^q x_3 \mid p+q=m+1\} \subseteq L^{\perp,2+3((m+1)-2)}. \tag{6.18}$$

Subcase B: i_0 is odd: In this case $m = 2(2k-1)+1 = 4k-1$, $i_0 = 2k-1$.

We may "repeat" the "Step 1" of the case "m **is even**" (corresponding to even i) to prove that the family

$$\left\{P_{m+1}\{x_1^{m-2j}x_2^{2j}x_3,\, x_1x_3\} \mid j = 0,\, 1,\, \dots,\, \frac{m-1}{2}\right\} \tag{6.19}$$

spans the subspace

$$span\left\{x_1^{m-(2j)}x_2^{2j+1}x_3 \mid j = 0,\, \dots,\, \frac{m-1}{2}\right\} \subseteq L^{\perp,2+3(m-2)+1}.$$

In the case i is odd, we replace the "bad" function corresponding to $0 = 2i_0 - m + 1 = 2(2k-1) - m + 1 = 4k - 1 - m$:

$$\begin{aligned}&P_{m+1}\{x_1^{m-(2k-1)}x_2^{2k-1}x_3,\, x_1x_3\}\\ &= \left(2(2k-1)x_1^{m-(2k-1)+2}x_2^{2k-2} + (2(2k-1)-m+1)x_1^{m-(2k-1)}x_2^{(2k-1)+1}\right)x_3\\ &= \left((m-1)x_1^{\frac{m+5}{2}}x_2^{\frac{m-3}{2}} + 0 \times x_1^{\frac{m+1}{2}}x_2^{\frac{m+1}{2}}\right)x_3,\end{aligned}$$

by the function

$$\begin{aligned}&P_{m+1}\{x_1^{m-2(k-1)}x_2^{2(k-1)+1}x_3,\, x_3\}\\ &= \left((2k-1)x_1^{m-2(k-1)+1}x_2^{2(k-1)} - (m-2(k-1))x_1^{m-2(k-1)-1}x_2^{2(k-1)+2}\right)x_3.\end{aligned}$$

This last function equals

$$\left(\frac{m-1}{2}x_1^{\frac{m+5}{2}}x_2^{\frac{m-3}{2}} - \frac{m+3}{2}x_1^{\frac{m+1}{2}}x_2^{\frac{m+1}{2}}\right)x_3 \in L^{\perp,2+3(m-2)+2}$$

and both the coefficients $\frac{m-1}{2}$ and $\frac{m+3}{2}$ are nonzero. We can conclude that the family

$$\left\{P_{m+1}\{x_1^{m-(2j-1)}x_2^{2j-1}x_3,\, x_1x_3\} \mid j = 1,\, 2,\, \dots,\, \frac{m+1}{2},\, 4j \neq m+1\right\}, \tag{6.20}$$

together with the new function and with

$$\frac{1}{1-m}\{\{x_1^m x_3,\, x_1x_3\},\, x_3\} = (x_1^{m+1} - m x_1^{m-1}x_2^2)x_2 \in L^{\perp,2+3(m-2)+2},$$

span the subspace

$$span\left\{x_1^{m-(2j-1)}x_2^{2j}x_3 \mid j = 0,\, 1,\, \dots,\, \frac{m+1}{2}\right\} \subseteq L^{\perp,2+3(m-2)+2}.$$

Therefore, if $m = 4j - 1$,

$$\{x_1^p x_2^q x_3 \mid p + q = m + 1\} \subseteq L^{\perp,2+3(m-2)+2} \subseteq L^{\perp,2+3((m+1)-2)}. \tag{6.21}$$

From (6.15), (6.18) and (6.21) we have that

$$\{x_1^p x_2^q x_3 \mid p+q=m+1\} \subseteq L^{\perp,2+3((m+1)-2)}$$

$\Big($for any $m \geq 2$ satisfying $\{x_1^p x_2^q x_3 \mid p+q=m\} \subseteq L^{\perp,2+3(m-2)}\Big)$.

By induction we conclude that for any $m \geq 2$ the monomials $x_1^p x_2^q x_3$ with $p+q=m$ are in $L^{\perp,2+3(m-2)}$; this is the statement of FOURTH STEP and, finishes the proof of the theorem. □

From the $l^\perp$ saturating set h given in theorem 6.4.9 we derive the l-saturating set $(\nabla^\perp\cdot)^{-1}h$ of solenoidal vector fields.

Corollary 6.4.10. *The set* $g = \left\{-\frac{1}{2}\vec{e_3},\ \frac{1}{6}(-x_1\vec{e_3}+x_3^2\partial_2),\ \frac{1}{12}\big(-(5x_1^2-1)\vec{e_3}+10x_1x_3^2\partial_2\big)\right\}$ *of vector eigenfunctions, is a l-saturating set in* $\mathbb{S}^2_+$.

Proof. We see that $s = \{\frac{1}{2}x_3,\ \frac{1}{6}x_1x_3,\ \frac{1}{12}(5x_1^2-1)x_3\}$ is the set of stream functions associated to the set of vorticities of theorem 6.4.9. Then the set of associated solenoidal vector fields is the set $(\nabla^\perp\cdot)^{-1}h = -\nabla^\perp s$. From the formula $\nabla^\perp f = x_3(\partial_1 f\partial_2 - \partial_2 f\partial_1)$ we obtain

$$\nabla^\perp x_3 = -x_1\partial_2 + x_2\partial_1 = \vec{e_3} \quad \text{and} \quad \nabla^\perp x_1 = x_3\partial_2,$$

where $\vec{e_3}$ is the vector field generating rotation on the Hemisphere (with angular velocity 1, around the axis x_3 of $\mathbb{R}^3$ and in the direction of $(0,\,0,\,1)\wedge(x_1,\,x_2,\,x_3)$).

From $\nabla^\perp fk = (\nabla^\perp f)k + f\nabla^\perp k$ we easily find that $g = -\nabla^\perp s$. □

Remark 6.4.2. *If we look at the proof of theorem 6.4.9 we have* $\{x_1^p x_2^q x_3 \mid p+q=m\} \subseteq L^{\perp,k} \Rightarrow \{x_1^p x_2^q x_3 \mid p+q=m+1\} \subseteq L^{\perp,k+2}$ *if* $m \neq 4k+1$ *and,* $\{x_1^p x_2^q x_3 \mid p+q=m\} \subseteq L^{\perp,k} \Rightarrow \{x_1^p x_2^q x_3 \mid p+q=m+1\} \subseteq L^{\perp,k+3}$ *if* $m = 4k+1$.

Therefore we have the better estimate $\{x_1^p x_2^q x_3 \mid p+q=m\} \subseteq L^{\perp,\,2(m-1)+(m-2)(\operatorname{div} 4)}$. *Where* $(m-2)(\operatorname{div} 4)$ *is the quotient of the entire division between* $m-2$ *and* 4, *i.e.,* $m-2 = (m-2)(\operatorname{div} 4)\times 4 + (m-2)(\operatorname{mod} 4)$.

Chapter 7

Controllability of Galerkin approximations

In chapter 6 we have given examples of saturating sets for the cases of the Torus, the Sphere and; the Rectangle and the Hemisphere under Lions boundary conditions. From those subspaces in the increasing sequence given by the definition of V-saturating set, we may select subspaces being the spanning of a finite set of eigenfunctions; in this case we may consider Galerkin approximations given by the cutting of big modes. We can prove time-t exact controllability of these approximations. We do it here for the case of the Rectangle; the case of the Torus is similar and we refer to [4]. From the proof we may guess we may proceed analogously in the cases of the Sphere and Hemisphere.

We make use of the terminology of the theories of Geometric Control and Lie Algebra; we assume some familiarity with those theories, if that is not the case we refer to the books [3] and [32].

7.1 The FCE procedure

In this section we present what we call the FCE procedure — a procedure with three steps: *Factorization+Convexification+Extraction.*

7.1.1 Factorization

Consider a control-affine system

$$\dot{q} = f(q) + \sum_{i=1}^{r} v_i(t) g_i(q) \quad q \in \mathbb{R}^n,\, v_i \in \mathbb{R} \tag{7.1}$$

where f, g_i are smooth vector fields and $[g_i, g_j] = 0$ for $i, j = 1, \ldots, r$; where $[\cdot, \cdot]$ denotes the Lie bracket.

In [3] it is proven that if we decompose the flow of system (7.1) as

$$\overrightarrow{exp} \int_0^t (f + gv(\tau)) d\tau := \overrightarrow{exp} \int_0^t (f + \sum_{i=1}^{r} v_i(t) g_i) d\tau$$

$$\overrightarrow{exp} \int_0^t (Ad\, G_v^\tau) f \, d\tau \circ G_v^t =: \overrightarrow{exp} \int_0^t (\mathrm{e}^{-gw(\tau)})_* f d\tau \circ G_v^t$$

where $g := (g_1, g_2, \cdots, g_r)$, $v := (v_1, v_2, \cdots, v_r)^T$, and G_v^t denotes the flow

$$\overrightarrow{exp} \int_0^t gv(\tau)d\tau = \mathrm{e}^{gw(t)}, \ w(t) = \int_0^t v(\tau)\, d\tau,$$

then

$$\overline{\mathcal{A}_{q_0}(f+gv)(t)} = \overline{\mathcal{A}_{q_0}((\mathrm{e}^{-gV})_* f)(t) \circ \{G_v^t \mid v(\tau) \in \mathbb{R}^r\}},$$

where $V \in \mathbb{R}^r$ is independent of v. Here $\mathcal{A}_x(y)(t)$ stays for the attainable set at time t from x following the vector fields y. Similarly, if we rewrite the system (7.1) as

$$\dot{q} = f(q) + \sum_{i=1}^{r} (v_i^1(\tau) + v_i^2(\tau)) g_i(q) \quad q \in \mathbb{R}^n, \, v_i^j \in \mathbb{R},$$

we arrive to

$$\overline{\mathcal{A}_{q_0}(f+gv)(t)} = \overline{\mathcal{A}_{q_0}((\mathrm{e}^{-gV^2})_* f_1)(t) \circ \{G_{v^2}^t \mid v^2(\tau) \in \mathbb{R}^r\}} \tag{7.2}$$

where $f_1(q) := f(q) + \sum_{i=1}^r v_i^1(\tau) g_i(q)$ and where v^1, v^2 and V^2 are independent. The system $\dot{q} = (\mathrm{e}^{-gV^2})_* f_1(q)$ is called *factorized system.*

Lemma 7.1.1. *With* $(\mathrm{e}^{-gV^2})_* f_1$ *and* $G_{v^2}^t$ *as in equation* (7.2) *there holds*

$$\overline{\mathcal{A}_{q_0}((\mathrm{e}^{-gV^2})_* f_1)(t) \circ \{G_{v^2}^t \mid v^2(\tau) \in \mathbb{R}^r\}} \supseteq \overline{\mathcal{A}_{q_0}((\mathrm{e}^{-gV^2})_* f_1)(t)} \circ \{G_{v^2}^t \mid v^2(\tau) \in \mathbb{R}^r\}.$$

Proof. Let $x \in \overline{\mathcal{A}_{q_0}((\mathrm{e}^{-gV^2})_* f_1)(t)} \circ \{G_{v^2}^t \mid v^2(\tau) \in \mathbb{R}^r\}$. Then there exist a point $y \in \overline{\mathcal{A}_{q_0}((\mathrm{e}^{-gV^2})_* f_1)(t)}$ and a control $u(\tau) \in \mathbb{R}^r$, $\tau \in [0, t]$ such that $x = y \circ G_u^t$. Let $y_n \longrightarrow y$, $\ y_n \in \mathcal{A}_{q_0}((\mathrm{e}^{-gV^2})_* f_1)(t)$. Hence $x_n = y_n \circ G_u^t$ is a sequence on $\mathcal{A}_{q_0}((\mathrm{e}^{-gV^2})_* f_1)(t) \circ G_u^t$ that converges to x. □

Therefore, system (7.1) is approximately controllable at time t if

$$\overline{\mathcal{A}_q((\mathrm{e}^{-gV^2})_* f_1)(t)} \circ \{G_{v^2}^t \mid v^2(\tau) \in \mathbb{R}^r\} = \mathbb{R}^n, \quad \forall q \in \mathbb{R}^n.$$

If g_i are constant vector fields, $g_i(q) = g_i$, $\quad i = 1, \ldots r$, $\quad q \in \mathbb{R}^n$, $\quad X \in \mathbb{R}^r$, they commute and the systems $\dot{q} = (\mathrm{e}^{-gX})_* f_1(q)$ and $\dot{q} = f_1(q + gX)$ coincide. A corollary of this is

Corollary 7.1.2. *System* (7.1) *(with g constant) is approximately controllable at time t if*

$$\overline{\mathcal{A}_q(f_{1X})(t)} \circ \{\mathrm{e}^{gV^2}\} = \mathbb{R}^n, \quad \forall q \in \mathbb{R}^n.$$

Here $X, V^2 \in \mathbb{R}^r, v^1(\tau) \in \mathbb{R}^r$ *and,*

$$\begin{aligned} f_{1X}(q) &:= f_1(q+gX) = f(q+gX) + g(q+gX)v^1 \\ &= f_X(q) + gv^1. \end{aligned}$$

In particular the system is approximately controllable at time t if

$$\overline{\mathcal{A}_q(f_{1X})(t)} = \mathbb{R}^n, \quad \forall q \in \mathbb{R}^n.$$

7.1.2 Convexification

If for some constant vector $\gamma \in \mathbb{R}^n$, $f(q)+\gamma$ belong to the convex set $Conv\{f_X \mid X \in \mathbb{R}^r\}$, then for every $u^1 \in \mathbb{R}^r$

$$f(q) + \gamma + gu^1 \in Conv\{f_X \mid X \in \mathbb{R}^r\} + span(g) \subseteq Conv\{f_X + gv^1 \mid X, v^1 \in \mathbb{R}^r\}.^1$$

This means that we can follow any of the vector fields $f(q) + \gamma + gv^1$ without changing the closure of the attainable set at time t (recall that convexification does not change the closure of attainable set at time t – see [32]). In particular system (7.1) is approximately controllable at time t if

$$\overline{\mathcal{A}_q(f(q) + \gamma + gv^1)(t)} = \mathbb{R}^n, \quad \forall q \in \mathbb{R}^n.$$

7.1.3 Extraction

Let C be a cone (including 0) and suppose that

$$f(q) + C \subseteq Conv\{f_X \mid X \in \mathbb{R}^r\}.$$

Then putting $G := span(g)$,

$$\begin{aligned} f(q) + C + G \subseteq f(q) + Conv(C) + G &\subseteq Conv\{f_X(q) + G \mid X \in \mathbb{R}^r\} \\ &= Conv\{f_{1X}(q) \mid X \in \mathbb{R}^r\}. \end{aligned}$$

Now from $Conv(C) + G$ we extract the linear space

$$G^1 := (G + Conv(C)) \cap (G - Conv(C)).$$

We shall call the directions from G^1 *"extracted" directions*. Since clearly $G \subseteq G^1$ because $0 \in C$, those directions in G will be called *"old" directions* and, those in $G^1 \backslash G$ *"new" directions*.

Adding new directions does not change the closure of attainable sets so, we can say that system (7.1) is approximately controllable at time t if the "bigger" system $\dot{q} = f(q) + g_1 v^1$ is, where $v^1 \in \mathbb{R}^{r_1}$, r_1 ($\geq r$) is the dimension of G^1 and g_1 is a matrix whose r_1 columns are vectors spanning G^1.

7.1.4 Iterating FCE's

Iterating FCE procedures we obtain an increasing sequence

$$G =: G^0 \subseteq G^1 \subseteq \ldots \subseteq G^j \subseteq \ldots$$

of subspaces of controlled directions without changing the closure of the attainable set at time t. Obviously if for some $p \in \mathbb{N}$ we have $G^p = \mathbb{R}^n$, then the approximate controllability at time t is an immediate consequence of corollary 7.1.2 (note that in such a case we can set for V^2 any vector from $\mathbb{R}^n$).

[1] Here $span(g)$ means the span of the columns of g. $Conv(A)$ stays for convexification of the set A.

7.2 Spectral method

We may write the equation as the infinite-dimensional ODE system:

$$\begin{aligned}\dot{u}_k := & \sum_{\substack{m,n\in\mathbb{N}_0^2\\ m<n\\ (n(++)m)^+=k}} u_m u_n C_{m,n}^{++} + \sum_{\substack{m,n\in\mathbb{N}_0^2\\ m<n\\ (n(--)m)^+=k}} u_m u_n C_{m,n}^{--}\\ & + \sum_{\substack{m,n\in\mathbb{N}_0^2\\ m<n\\ (n(-+)m)^+=k}} u_m u_n C_{m,n}^{-+} + \sum_{\substack{m,n\in\mathbb{N}_0^2\\ m<n\\ (n(+-)m)^+=k}} u_m u_n C_{m,n}^{+-}\\ & - \nu\bar{k}u_k + F_k + v_k = -B_k(u,\, u) - \nu\bar{k}u_k + F_k + v_k; \end{aligned} \tag{7.3}$$

where $-B_k(u,\, u)$ denotes the k^{th} coordinate of $-Bu = -P^{\nabla}[(u\cdot\nabla)u]$.

Definition 7.2.1. *A $\mathcal{G}$-**Galerkin approximation** of system* (7.3) *is the same system with the additional condition $k,\, n,\, m\, \in \mathcal{G} \in \mathcal{FP}(\mathbb{N}_0^2)$.*

By $\mathcal{FP}(\mathbb{N}_0^2)$ we mean the set composed by the finite subsets of $\mathbb{N}_0^2$.

As in chapter 6, put for each $M \in \mathbb{N}_0$:

$$\begin{aligned}\mathcal{K}^M &:= \{(n_1,\, n_2) \in \mathbb{N}_0^2 \mid n_1,\, n_2 \le M+2\} \setminus \{(M+2, M+2)\};\\ \kappa_M &:= \#\mathcal{K}^M = (M+2)^2 - 1; \quad g^{M-1} = \{W_n \mid n \in \mathcal{K}^M\}; \quad G^{M-1} = span\{g^{M-1}\}.\end{aligned}$$

Recall that

$$W_k := \begin{pmatrix} \frac{-k_2\pi}{b}\sin\left(\frac{k_1\pi x_1}{a}\right)\cos\left(\frac{k_2\pi x_2}{b}\right)\\ \frac{k_1\pi}{a}\cos\left(\frac{k_1\pi x_1}{a}\right)\sin\left(\frac{k_2\pi x_2}{b}\right)\end{pmatrix}, \qquad k = (k_1,\, k_2) \in \mathbb{N}_0^2.$$

are eigenfunctions of the Laplace-de Rham operator with $\Delta W_k = \bar{k}W_k$ and $\bar{k} := \pi^2\left(\frac{k_1^2}{a^2} + \frac{k_2^2}{b^2}\right)$.

Write g for the matrix whose columns are the $\kappa_1 = 8$ vectors in g^0.

Write the $\mathcal{K}^N$-Galerkin approximation of system (7.3), with the directions in g^0 as the controlled ones, as

$$\dot{u} = f(u) + gv, \qquad v \in \mathbb{R}^8,\ u \in G^{N-1}$$

or, equivalently

$$\dot{u}_k = (f(u))_k + (gv)_k, \qquad v \in \mathbb{R}^8,\ u \in G^{N-1},\ k \in \mathcal{K}^N \tag{7.4}$$

where $f(u) = -Bu - \nu Au + F^{\kappa_N}$ and, F^{κ_N} is the projection of F onto G^{N-1}; $A = \Delta$.

Let us apply a FCE procedure to this finite-dimensional system. After factorization we obtain the factorized system

$$\begin{aligned}&(f_{1X}(u))_k = (f_X(u))_k + gv^1\\ =&(f(u))_k - B_k(u,\, gX) - B_k(gX,\, u) - \nu\bar{k}(gX)_k - B_k((gX),\, (gX)) + gv^1\end{aligned}$$

Now we put

$$\begin{aligned}(\mathcal{V}_0(gX))_k &:= (f(u))_k + gv^1;\\ (\mathcal{V}_1(gX))_k &:= -B_k(u,\, gX) - B_k(gX,\, u) - \nu\bar{k}(gX)_k;\\ (\mathcal{V}_2(gX))_k &:= -B_k((gX),\, (gX));\end{aligned}$$

and we note that $\mathcal{V}_0$, $\mathcal{V}_1$, and $\mathcal{V}_2$ are respectively, independent, linear and bilinear on the vector field gX.

Now, given $X \in \mathbb{R}^r$ we have

$$\mathcal{V}_0(g(-X)) = \mathcal{V}_0(gX); \quad \mathcal{V}_1(g(-X)) = -\mathcal{V}_1(gX) \text{ and } \mathcal{V}_2(g(-X)) = \mathcal{V}_2(gX)$$

so,

$$f(u) - B(gX,\, gX) = \frac{1}{2}\Big(f_X(u) + f_{-X}(u)\Big) \in Conv\{f_X(u) \mid X \in \mathbb{R}^r\}.$$

Observe that g^0 is known to be V-saturating, then after N iterations of the FCE procedure we conclude that we may use controls in G^{N-1} without changing the closure of the attainable set at time t of system (7.4): in the first iteration we may extract vectors spanning the space $span\{g^1 \backslash \{W_{(3,\,3)}\}\}$; in the second we may extract vectors spanning $span\{g^1 \cup \{W_{(1,\,5)},\, W_{(3,\,5)}\}\}$ and; in the j^{th} iteration, $j \geq 3$, we may extract vectors spanning $span\{g^{j-1}\} = G^{j-1}$.

Then we conclude:

Corollary 7.2.1. *The $\mathcal{K}^N$-Galerkin approximation*

$$\dot{u}_k = (f(u))_k + (g^0 v)_k, \qquad v \in \mathbb{R}^8,\ u \in G^{N-1},\ k \in \mathcal{K}^N \tag{7.5}$$

is approximately controllable at time t. g^0 denotes the matrix whose columns are the 8 elements of g^0.

7.3 Exact controllability of Galerkin approximations

Write the $\mathcal{K}^N$-Galerkin approximation of the Navier-Stokes system (7.3), with $\mathcal{K}^1$ as the set of excited modes, in the concise form

$$N: \begin{cases} \dot{u}_k = -B_k(u) - \nu A u_k + F_k + v_k & k \in \mathcal{K}^1 \\ \dot{u}_k = -B_k(u) - \nu A u_k + F_k & k \in \mathcal{K}^N \setminus \mathcal{K}^1 \\ u \in \mathbb{R}^{\kappa_N}. \end{cases} \tag{7.6}$$

In [23] W. E and J. Mattingly proved the *full Lie rank property* for the 2D Navier-Stokes equation with periodic conditions and for some class of few low modes controls. Now we prove that also, in the present case of the Rectangle under Lions boundary conditions, our equation is full Lie rank, i.e., Lie brackets at each point span the ambient space $\mathbb{R}^{\kappa_N}$.

Before we have proved that for all $N \in \mathbb{N}_0$ and all $t > 0$ the system [(7.6).N] is time-t approximately controllable:

$$\forall u \in \mathbb{R}^{\kappa_N} \quad \overline{\mathcal{A}_u(\mathcal{F}_N)(t)} = \mathbb{R}^{\kappa_N}$$

where $\mathcal{F}_N$ is the family of vector fields of system [(7.6).N], i.e.,

$$\mathcal{F}_N = \{-B(\cdot) - \nu A(\cdot) + F^{\kappa_N} + v \mid v \in \mathbb{R}^{\kappa_1}\}.$$

Next we prove the (exact) controllability of system [(7.6).N], i.e., $\mathcal{A}_u(\mathcal{F}_N)(t) = \mathbb{R}^{\kappa_N}$ for all $u \in \mathbb{R}^{\kappa_N}$. For that we need to compute some Lie brackets.

Lie brackets. Full Lie rank property.

Set the vector fields

$$V^0 := -B - \nu A + F^{\kappa_N}, \quad X^i(\pm) := V^0 \pm \frac{\partial}{\partial u_i}.$$

By Induction we prove that all constant vector fields $\frac{\partial}{\partial_{u_i}}$, $i \in \mathcal{K}^N$ are linear combination of brackets.

If $i \in \mathcal{K}^1$ we have $\frac{\partial}{\partial_{u_i}} = \frac{1}{2}X(+) - \frac{1}{2}X(-)$.

Inductive step: Suppose that, for all $p < N$ and $i \in \mathcal{K}^p$, $\frac{\partial}{\partial_{u_i}}$ is a linear combination of brackets in $\{b_j \mid j = 1, \ldots, M\}$. Then

$$V^i(\pm) := [X^i(\pm), V^0] = \pm \left[\frac{\partial}{\partial u_i}, V^0\right] = \pm \frac{\partial V^0}{\partial u}\frac{\partial}{\partial u_i} = \pm \frac{\partial V^0}{\partial u_i},$$

so, $V^i(\pm) = \sum_{k=1}^{\kappa_N} V_k^i(\pm)\frac{\partial}{\partial u_k}$ with

$$V_k^i(\pm) := \pm \left[\sum_{\substack{k=(n++i)^+ \\ i<n}} u_n C_{i,n}^{++} + \sum_{\substack{k=(n+-i)^+ \\ i<n}} u_n C_{i,n}^{+-} \right.$$

$$\left. + \sum_{\substack{k=(n-+i)^+ \\ i<n}} u_n C_{i,n}^{-+} + \sum_{\substack{k=(n--i)^+ \\ i<n}} u_n C_{i,n}^{--} \right] \mp \delta_{k,i}\nu\bar{k},$$

where $\delta_{k,i}$ is the Kronecker delta function.

Now we compute also

$$V^{j,i}(\pm) := [X^j(\pm), V^i(+)] = [V^0, V^i(+)] \pm \frac{\partial V^i(+)}{\partial u_j},$$

and obtain,

$$V^{j,i}(\pm) = [V^0, V^i(+)] \pm \gamma_{i,j}, \quad j > i;$$

with

$$\gamma_{i,j} = C_{i,j}^{++}\frac{\partial}{\partial_{u_{(j(++)i)^+}}} + C_{i,j}^{+-}\frac{\partial}{\partial_{u_{(j(+-)i)^+}}} + C_{i,j}^{-+}\frac{\partial}{\partial_{u_{(j(-+)i)^+}}} + C_{i,j}^{--}\frac{\partial}{\partial_{u_{(j(--)i)^+}}}.$$

Thus $\gamma_{i,j} = \frac{1}{2}V^{j,i}(+) - \frac{1}{2}V^{j,i}(-)$, with $j > i$, is a linear combination of brackets. Therefore also every $\frac{\partial}{\partial u_n}$, $\quad n \in \mathcal{K}^{p+1}$ is a combination of brackets because we already know that they are combinations of the $\gamma_{i,j}$

Therefore, for all $N \in \mathbb{N}_0$, system [(7.6).N] is a full-rank bracket generating system. From that and from its approximate controllability [2] we conclude its controllability. Unfortunately for fixed time the bracket generating property is not sufficient to conclude controllability from approximate controllability. To achieve controllability at time t we shall need some lemmas which proofs can be found in [32].

[2] Approximate controllability at time t trivially implies approximate controllability. By controllability it is usually meant that given two state points, we can drive the system from one to the other in finite time; that finite time may depend on the referred pair of state points.

7.3.1 Zero orbits and zero ideal

Definition 7.3.1. *A* **zero-time orbit** N_{0u} *through* u *of a family of vector fields* $\mathcal{F}$ *is the set*

$$N_{0u} := \{u \circ e^{t_1 V_1} \circ \ldots \circ e^{t_p V_p} \mid p \in \mathbb{N}_0,\, V_i \in \mathcal{F},\, t_i \in \mathbb{R},\, \sum_{i=1}^{p} t_i = 0\}.$$

Definition 7.3.2. *The* **derived algebra** *of* $\mathcal{F}$*, denoted* $\mathcal{D}_{er}(\mathcal{F})$*, is the set of all linear combinations of iterated brackets.*[3]

The **zero-time ideal**, *denoted* $\mathcal{I}(\mathcal{F})$*, is the spanning of elements in* $\mathcal{D}_{er}(\mathcal{F})$ *and differences of the form* $X - Y$ *with* X *and* Y *in* $\mathcal{F}$*.*

Lemma 7.3.1. *Let* $\mathcal{F}$ *be any family of analytic vector fields on an analytic manifold* M*. Let* N *be an orbit of* $\mathcal{F}$ *and,* N_0 *be a zero orbit of* $\mathcal{F}$ *contained in* N*. Then we have the following:*

- *Each connected component of* N_0 *is an orbit of* $\mathcal{I}(\mathcal{F})$*;*
- *For each* $u \in N_0$*, the tangent space of* N_0 *at* u *is equal to the evaluation of* $\mathcal{I}(\mathcal{F})$ *at* u*;*
- *The dimension of* $\mathcal{I}_u(\mathcal{F})$ *is constant as* u *varies on* N*. It is equal either to* $dim(Lie_u(\mathcal{F}))$ *or to* $dim(Lie_u(\mathcal{F})) - 1$*;*
- $dim(Lie_u(\mathcal{F})) = dim(\mathcal{I}_u(\mathcal{F}))$ *if, and only if,* $X(u) \in \mathcal{I}_u(\mathcal{F})$ *for some* $X \in \mathcal{F}$*.*

Lemma 7.3.2. *Suppose that* $\mathcal{F}$ *is a family of vector fields on* M *such that both* $\mathcal{F}$ *and its zero-time ideal* $\mathcal{I}(\mathcal{F})$ *are Lie-determined (the evaluation of Lie brackets at each point span the tangent space to the orbit). In addiction, assume that* $\mathcal{F}$ *contains a complete vector field. Then*

- $\mathcal{A}_u(\mathcal{F})(t)$ *is a connected subset of some zero orbit* N_{0z} *through some element* $z \in M$*.*
- $\mathcal{A}_u(\mathcal{F})(t)$ *has a nonempty interior in the manifold topology of the zero-orbit where it is contained. Moreover, the set of interior points is dense in* $\mathcal{A}_u(\mathcal{F})(t)$*.*

Coming back to our system [(7.6).N], we observe that $V^0(0) = F^{\kappa_N}$ is a constant vector field; that $\frac{\partial}{\partial_{u_n}} \in \mathcal{D}_{er}(\mathcal{F}_N) \subseteq \mathcal{I}(\mathcal{F}_N)$ for $n \in \mathcal{K}^N \setminus \mathcal{K}^1$ and; that $\frac{\partial}{\partial_{u_i}} = \frac{1}{2}[X^i(+) - X^i(-)] \in \mathcal{I}(\mathcal{F}_N)$ for $i \in \mathcal{K}^1$. Since F^{κ_N} is a linear combination of the $\frac{\partial}{\partial_{u_n}}$, $n \in \mathcal{K}^N$, we have that $V^0(0) = F^{\kappa_N} \in \mathcal{I}_0(\mathcal{F}_N)$.

By lemma 7.3.1 we have

$$dim(Lie_0(\mathcal{F}_N)) = dim(\mathcal{I}_0(\mathcal{F}_N)) = \kappa_N;$$

which means that the zero-time orbit N_0 through 0 has dimension κ_N and, since that dimension is constant in all points in the unique orbit $\mathbb{R}^{\kappa_N}$ of the system, we conclude that N_0 is a union of connected components of dimension κ_N. Since the dimension of that components is κ_N their topology coincide with that of $\mathbb{R}^{\kappa_N}$ and, from the fact that the zero-time orbits form a partition of $\mathbb{R}^{\kappa_N}$ we conclude that $\mathbb{R}^{\kappa_N}$ is a union of connected open sets. Therefore there is only one zero-orbit, it is the whole state space $\mathbb{R}^{\kappa_N}$.

[3]Brackets of "length" ≥ 1, considering the elements of $\mathcal{F}$ as brackets of length 0.

By lemma 7.3.2, and by the fact that V^0 is a complete vector field which follows from the estimate $|u(s)| \leq |u(0)| + \frac{s}{\nu}\|F\|^2_{V'}$ (see estimate (1.20)), the interior $int\mathcal{A}_u(\mathcal{F}_N)(t)$ of the attainable set from u at time t is dense in $\mathcal{A}_u(\mathcal{F}_N)(t)$, where the interior and density are relative to the topology of $\mathbb{R}^{\kappa_N}$ because that is the topology of the zero-orbit. Hence we arrive to the equality

$$\overline{int\mathcal{A}_u(\mathcal{F}_N)(t)} = \overline{\mathcal{A}_u(\mathcal{F}_N)(t)} = \mathbb{R}^{\kappa_N}$$

for all $t > 0$.

Now we can prove the controllability at time t of system [(7.6).N]: Let u, z be two elements in $\mathbb{R}^{\kappa_N}$. Since the intersection of two open dense sets stills open and dense, we may take a point

$$w \in int\mathcal{A}_u(\mathcal{F}_N)(t/2) \cap int\mathcal{A}_z(-F_N)(t/2).$$

Note that the family $-\mathcal{F}_N := \{-V \mid V \in \mathcal{F}_N\}$ satisfies the requirements of lemmas 7.3.1 and 7.3.2 because $\mathcal{F}_N$ does.

Then we can write

$$w = u \circ e^{t_1 V_1} \circ \cdots \circ e^{t_n V_n}, \qquad V_i \in \mathcal{F}_N,\, t_i \geq 0,\, \sum_{i=1}^{n} t_i = \frac{t}{2};$$

$$w = z \circ e^{-s_1 W_1} \circ \cdots \circ e^{-s_m W_m}, \qquad W_i \in \mathcal{F}_N,\, s_i \geq 0,\, \sum_{i=1}^{m} t_i = \frac{t}{2};$$

So, z is reachable from u in time t:

$$z = u \circ e^{t_1 V_1} \circ \cdots \circ e^{t_n V_n} \circ e^{s_m W_m} \circ \cdots \circ e^{s_1 W_1}.$$

Chapter 8

Perturbation of the metric on a compact Riemannian manifold

We consider the Navier-Stokes system on a compact analytic two-dimensional Riemannian manifold M either boundaryless or simply-connected under Lions boundary conditions. Recall that a vector field V satisfies Lions boundary conditions if both $g(V, \mathbf{n})$ and $\nabla^\perp \cdot V$ vanish on the boundary; g being the metric tensor.

We connect two given analytic metrics in M, by an analytic homotopy in the space of metrics, and study how to derive results on controllability known for one metric to the other.

Given a special $l^\perp$-saturating set for some metric M, then for many other metrics on M we prove that we can observe solid controllability in those subspaces coinciding with the spanning of a finite number of eigenfunctions of the Laplacean.

We follow the idea presented in [6]. The meaning of *"many other"* will be clear below but, for the moment we simply say that, in some sense, it means *"a dense set of"*.

8.1 Connection of metrics on M

Consider an analytic metric $\mu(0) \in T^*M \otimes T^*M$ on an analytic compact manifold M. We consider the cases M is either boundaryless or simply-connected under Lions boundary conditions.

We assume that M together with its boundary ∂M are contained in a bigger manifold $\tilde{M}$, and that the analytic function $\mu(0)$ is analytic in $\tilde{M}$.

As a consequence, if we write $\mu(0)$ in local coordinates $\mu(0) = g_{ij}(0)dx^i \otimes dx^j$, by the compactness of our manifold we may suppose that $\sqrt{\bar{g}(0)} = \det[g_{ij}(0)]$ is bounded from below in $\overline{M}$ by a positive constant b_0, i.e., for all points of $\overline{M}$ and any chart containing it we have $\sqrt{\bar{g}(0)(x^1, x^2)} > b_0$.

Given another metric $\mu(1) \in T^*M \otimes T^*M$ on M, we connect $\mu(0)$ to $\mu(1)$ by an analytic "homotopy" $H(t)$ in the space of metrics. That is possible, for example by $H(t) = (1 - t)\mu(0) + t\mu(1)$; symmetry, bilinearity and positive definiteness of $H(t)$ at a given fiber T_xM of the tangent bundle follow from the same properties we have for $\mu(0)$ and $\mu(1)$.

From now we denote a given homotopy $H(t)$ in the space of metrics in M, connecting the metrics $\mu(0)$ and $\mu(1)$, by $\mu(t)$.

The Laplacean $\Delta(t)f = - *_t d *_t df$ of a function f, defined in $(M, \mu(t))$, depends analytically on the parameter $t \in [0, 1]$, where $*_t$ is the Hodge map in $(M, \mu(t))$. We suppose

$\sqrt{\bar{g}(t)}$ are bounded below by b_0 and above by B_0, where b_0 and B_0 are positive constants, independent of $t \in [0, 1]$. Locally, since the homotopy is analytic, the coefficients $g_{ij}(t)(x^1, x^2)$ of the metric tensor $\mu(t)$ depend analytically on the real variables t, x^1 and x^2 and, we recall that the Laplacean is given by

$$-\frac{1}{\sqrt{\bar{g}(t)}}\partial_i(\sqrt{\bar{g}(t)}g^{ij}(t)\partial_j f).$$

For each $t \in [0, 1]$, let $\{E(t)_n \mid n \in \mathbb{N}_0\}$ be the complete system of eigenfunctions of $\Delta(t)$ in $L^2(M, \mu(t))$ and, let $\{\lambda(t)_n \mid n \in \mathbb{N}_0\}$ be the corresponding (repeated) eigenvalues:

$$\Delta(t)E(t)_n = \lambda(t)_n E(t)_n \; in \; M; \qquad \begin{cases} E(t)_n = 0 \; on \; \partial M & if \; \partial M \neq \emptyset \\ \int_M E(t)_n \sqrt{\bar{g}(t)} dx^1 \wedge dx^2 = 0 & if \; \partial M = \emptyset \end{cases}.$$

The system $\{E(t)_n, n \in \mathbb{N}_0\}$ is also complete on $L^2(M, \mu(0))$ because, for any $g \in L^2(M, \mu(0))$, if $\forall n \int_M gE(t)_n \sqrt{\bar{g}(0)}\, dx^1 \wedge dx^2 = 0$, then

$$\forall n \quad \int_M gE(t)_n \frac{\sqrt{\bar{g}(0)}}{\sqrt{\bar{g}(t)}} \sqrt{\bar{g}(t)}\, dx^1 \wedge dx^2 = 0,$$

which implies that $g\frac{\sqrt{\bar{g}(0)}}{\sqrt{\bar{g}(t)}} = 0$, i.e., $g = 0$.

Besides if the family $\{E(t)_n \mid n \in \mathbb{N}_0\}$ is taken orthonormal in $L^2(M, \mu(t))$ then the family $\left\{\sqrt[4]{\frac{\bar{g}(t)}{\bar{g}(0)}}E(t)_n \mid n \in \mathbb{N}_0\right\}$ is orthonormal in $L^2(M, \mu(0))$.

Since $L^2(M, \mu(t)) = L^2(M, \mu(0))$, for all $t \in [0, 1]$ we may see $\Delta(t)$ as an operator on $L^2(M, \mu(0))$.

The Poisson bracket $\{f, g\}_t$ between two functions f, g defined on $(M, \mu(t))$ is given by the relation $*_t(df \wedge dg)$ so, locally

$$\{f, g\}_t = \frac{\partial_1 f \partial_2 g - \partial_2 f \partial_1 g}{\sqrt{\bar{g}(t)}}$$

and we arrive to the identity

$$\{f, g\}_t = \frac{\sqrt{\bar{g}(0)}}{\sqrt{\bar{g}(t)}}\{f, g\}_0;$$

as we may see, $\{f, g\}_t$ depends analytically on the parameter t.

To study the existence of $l^\perp$-saturating sets we need to iterate the operation

$$D(t)_g(\cdot) = \{\Delta(t)^{-1}(\cdot), g\}_t + \{\Delta(t)^{-1}g, (\cdot)\}_t$$

on $(M, \mu(t))$ that, when brought to $(M, \mu(0))$, reads

$$D(t)_g(\cdot) = \frac{\sqrt{\bar{g}(0)}}{\sqrt{\bar{g}(t)}} \left\{\Delta(t)^{-1}(\cdot), g\}_0 + \{\Delta(t)^{-1}g, (\cdot)\right\}_0$$

seeing $\Delta(t)$ as an operator in $L^2(M, \mu(0))$.

Such operation depends analytically in t. Moreover locally, given functions f, g that are analytic in $(t, x^1, x^2) \in [0, 1] \times \overline{M}$ also $D(t)_g f$ is analytic because partial derivative of analytic functions are analytic and preserve the radius of convergence of the power series at a given point (see [15, section IV§1.3]). Note that if $\Delta(t)^{-1}f$ was not analytic at a given point then $f = - *_t d *_t d\Delta^{-1}(0)f$ would not be analytic at the same point, because both $*_t w$ and dw are analytic if, and only if, the form w is analytic.

8.2 Analytic perturbation of linear operators

In this section we collect some classical results, on analytic perturbation theory for linear operators, from the Kato's book [33].

8.2.1 Finite-dimensional case

By classical result of perturbation theory (see [33, ch. II]) the system of eigenvalues of a family of linear operators $T(\varkappa)$ analytic in a domain $\varkappa \in D_0 \subseteq \mathbb{C}$, are (branches of) analytic functions. Possible singularities of these analytic functions are algebraic so that, beyond some *exceptional* points in D_0, the eigenvalues are given by d analytic functions $\lambda(\varkappa)_j$, $j = 1, \ldots, d$. The number of exceptional points are finite in each compact subset of D_0.

Following [33, section II§1.4] we write $R(\zeta, \varkappa) := (T(\varkappa) - \zeta)^{-1}$ for the resolvent of $T(\varkappa)$ and $\Sigma(T(\varkappa))$ for the set of all eigenvalues of $T(\varkappa)$, called spectrum of $T(\varkappa)$. The domain of the resolvent $R(\zeta, \varkappa)$ is called the resolvent set and is given by $P(T(\varkappa)) = D_0 \setminus \Sigma(T(\varkappa))$. The operator

$$P(\varkappa) = -\frac{1}{2\pi i} \int_{\Gamma(\varkappa)} R(\zeta, \varkappa)\, d\zeta$$

is the sum of the eigenprojections for all eigenvalues of $T(\varkappa)$ lying inside the positively oriented closed curve Γ contained in the resolvent set.

Consider a simply connected domain $D \subseteq D_0$ containing no non-exceptional points. The eigenprojection $P(\varkappa)_j = -\frac{1}{2\pi i} \int_{\Gamma(\varkappa)_j} R(\zeta, \varkappa)\, d\zeta$ associated with the eigenvalue $\lambda(\varkappa)_j$ (where $\Gamma(\varkappa)_j$ is a curve in $P(T(\varkappa))$ enclosing the eigenvalue $\lambda(\varkappa)_j$ and no other) is holomorphic in D and the multiplicity of each $\lambda(\varkappa)_j$ is constant in D.

Summarizing (see [33, ch. II, th. 1.8]):

Theorem 8.2.1. *The eigenvectors $\lambda(\varkappa)_j$ and the eigenprojections $P(\varkappa)_j$ of $T(\varkappa)$ are (branches of) analytic functions for $\varkappa \in D_0$ with only algebraic singularities at some (but not necessarily all) exceptional points. $\lambda(\varkappa)_j$ and $P(\varkappa)_j$ have all branch points in common.*

Moreover (see [33, ch. II, th. 1.9]]):

Theorem 8.2.2. *If $\varkappa = \varkappa_0$ is a branch point for $\lambda(\varkappa)_j$ (and therefore also for $P(\varkappa)_j$), then $P(\varkappa)_j$ has a pole there. In particular the norm $\|P(\varkappa)_j\|$ goes to ∞ as $\varkappa$ goes to $\varkappa_0$.*

As a corollary (see [33, ch. II, th. 1.9]]):

Theorem 8.2.3. *Let $\varkappa_0 \in D_0$ (possibly an exceptional point) and let there exist a sequence $\varkappa^n$ converging to $\varkappa_0$ such that $\|P(\varkappa^n)_j\|$ is bounded by some constant (independent of n). Then all the $\lambda(\varkappa)_j$ and $P(\varkappa)_j$ are holomorphic at $\varkappa = \varkappa_0$.*

As soon as we have a family of projections $P(\varkappa)_j$, $\quad j = 1, \ldots, d$ depending holomorphically on $\varkappa \in D$, where D is simply connected (and, without loss of generality, we suppose to contain 0), it is possible to construct a so-called *transformation function* $U(\varkappa)$, for those projections, satisfying

1. The inverse $U^{-1}(\varkappa)$ exists and both $U(\varkappa)$ and $U^{-1}(\varkappa)$ are holomorphic for $\varkappa \in D$;

2. $U(\varkappa)P(0)_j U^{-1}(\varkappa) = P(\varkappa)_j, \quad j = 1, \ldots, d;$
3. Given a basis $\phi(0)_{j1}, \ldots, \phi(0)_{jm_j}$ for the image $M(0)_j = \mathsf{R}(P(0)_j)$ of the projection $P(0)_j$; the set $\phi(\varkappa)_{j1}, \ldots, \phi(\varkappa)_{jm_j}$, where $\phi(\varkappa)_{jk} := U(\varkappa)\phi(0)_{jk}$, form a basis for the image $M(\varkappa)_j = \mathsf{R}(P(\varkappa)_j)$ of the projection $P(\varkappa)_j$.

In particular, for each pair (j, k) the vectors $\phi(\varkappa)_{jk}$ are holomorphic in $\varkappa$.

8.2.2 Infinite-dimensional case

All the results for the finite dimensional case are still valid in the infinite dimensional case as soon we are concerned with a finite number of eigenvalues (and respective eigenprojections) (see [33, section VII§1.3]).

8.3 Controllability

From now we work under the following hypothesis:

Hypothesis 1. *There is a finite $l^{\perp}$-saturating set $h = \{f(0)_1, f(0)_2, \ldots, f(0)_s\}$ for $(M, \mu(0))$ consisting of s eigenfunctions of the Laplacean $\Delta(0)$ on $(M, \mu(0))$.*

Definition 8.3.1. *We call a finite-dimensional subspace $\mathcal{L} \subseteq H$ of divergence free vector fields a* **coordinate space** *if it is the spanning of a finite number of eigenfunctions of eigenfunctions of eigenfunctions of the Laplacean, under Lions boundary conditions.*

Definition 8.3.2. *We say that the Navier-Stokes system on M is* **time-T solidly controllable on observed coordinate space** *if it is time-T solidly controllable on the observed component $\mathcal{L}$, for all coordinate space $\mathcal{L}$.*

As we see, the notion of controllability on observed coordinate space is a particular case of controllability on observed component defined in chapter 3. Anyway, form the practical point of view, coordinate spaces are perhaps, the more interesting to observe because, somehow we observe better the transferring of energy between modes.

Definition 8.3.3. *We call a subset of topological space $\mathcal{T}$* **residual** *if it contains an intersection of countable family of open dense subsets of $\mathcal{T}$.*

Theorem 8.3.1. *Under hypothesis 1; for all compact Riemannian manifolds M (either boundaryless or with analytic boundaries and under Lions boundary conditions) there exist a residual set $\mathcal{R}_\mu \subset [0, 1]$ and, for $t \in \mathcal{R}_\mu$, s eigenfunctions (modes) $f(t)_1, \ldots, f(t)_s$ of the Laplace operator $\Delta(t)$ on $(M, \mu(t))$ such that the Navier-Stokes system on $(M, \mu(t))$ is controllable on observed coordinate space by means of (controlled) forcing applied to the modes in the finite set $(\nabla^{\perp}\cdot)^{-1}\{f(t)_1, \ldots, f(t)_s\}$.*

This theorem will be proven below. In particular the theorem says that there are metrics $\mu(t)$ for t close to 1, for which we can observe controllability in each coordinate space.

We extend the family of "Laplaceans" $\Delta(t)$ defined in $L^2(M, \mu(0))$ analytically to a neighborhood of the segment $[0, 1]$ in the complex plane: locally for $\varkappa$ in that neighborhood $\Delta(\varkappa)$ is defined in the space $L^2(M, \mu_0) + iL^2(M, \mu_0)$ by

$$-\frac{1}{\sqrt{\bar{g}(\varkappa)}}\partial_i(\sqrt{\bar{g}(\varkappa)}g^{ij}(\varkappa)\partial_j f). \tag{8.1}$$

where $g^{ij}(\varkappa)$, and is an analytic extension of $g^{ij}(t)$.

From the classical results presented in section 8.2 we derive that a finite number of eigenvalues and eigenprojections of the operator $\Delta(\varkappa)$, for real $\varkappa$ depend analytically in $\varkappa$. We have to prove that there are no singularities, i.e., that the projections onto eigenspaces do not explode: for real t consider the orthonormal complete system $\left\{\sqrt[4]{\frac{\bar{g}(t)}{\bar{g}(0)}}E(t)_n \mid n \in \mathbb{N}_0\right\}$ in $L^2(M,\, \mu(0))$; any vector w is written in a unique way as $w = \sum_{k\in\mathbb{N}} w_k E(t)_k$, where $w_k = \int_M wE(t)_k\sqrt{\bar{g}(t)}\, dx^1 \wedge dx^2 = \int_M w\sqrt[2]{\frac{\bar{g}(t)}{\bar{g}(0)}}E(t)_k\sqrt{\bar{g}(0)}\, dx^1 \wedge dx^2$.

Thus the "projection onto each one-dimensional eigenfunction space" associated with $E(t)_k$ has a norm bounded by

$$\left|\sqrt[2]{\frac{\bar{g}(t)}{\bar{g}(0)}}E(t)_k\right|_{L^2(M,\,\mu(0))} \le \left(\max_{[0,\,1]\times\overline{D}}\left|\sqrt[4]{\frac{\bar{g}(t)}{\bar{g}(0)}}\right|\right)\left|\sqrt[4]{\frac{\bar{g}(t)}{\bar{g}(0)}}E(t)_k\right|_{L^2(M,\,\mu(0))},$$

i.e., the norm of the projection is bounded by $\max_{[0,\,1]\times\overline{D}}\left|\sqrt[4]{\frac{\bar{g}(t)}{\bar{g}(0)}}\right|$ which is bounded by some constant M because, $\bar{g}(t)$ is bounded and $\bar{g}(0)$ is bounded from below by a positive constant b_0. The bound for the norm of the projection is independent of t and k.

Also the eigenprojection $P(t)_j$, being the finite sum of those projections corresponding to the eigenfunctions associated with the eigenvalue $\lambda(t)_j$, do not explode for real t.

So $\Delta(\varkappa)$ can not have a pole in any point of the real line and, we may conclude that a finite system of eigenvalues and of eigenfunctions of the operator $\Delta(t)$ depend analytically on t for real t.

Remark 8.3.1. *Writing $u = \sum_{k\in\mathbb{N}} u_k E(t)_k$ the operator $\Delta(t) - \zeta$ may be seen as a triangular matrix $\mathbb{T}(\delta)$ which diagonal is $\delta = (\lambda(t)_0 - \zeta,\, \lambda(t)_1 - \zeta,\, \dots,\, \lambda(t)_n - \zeta,\, \dots)$, where $\lambda(t)_k$ is the eigenvalue associated to the eigenfunction $E(t)_k$ (there may be repetition of eigenvalues). Thus its inverse is given by*

$$(\Delta(t) - \zeta)^{-1} = \mathbb{T}((\lambda(t)_0 - \zeta)^{-1},\, (\lambda(t)_1 - \zeta)^{-1},\, \dots,\, (\lambda(t)_n - \zeta)^{-1},\, \dots).$$

The integral $\frac{1}{2\pi i}\int_C (z - z_0)^{-1}\, dz$, is the so-called index $I(C,\, z_0)$ of the closed curve C relative to z_0; it is well known (see [15, section II§1.8]) that $I(C, z_0) = 1$ if z_0 is enclosed by C and $I(C, z_0) = 0$ if z_0 is exterior to C. So we may rewrite

$$P(t)_j = \mathbb{T}(I(\Gamma(t)_j,\, \lambda(t)_0),\, I(\Gamma(t)_j,\, \lambda(t)_1),\, \dots,\, I(\Gamma(t)_j,\, \lambda(t)_n),\, \dots);$$

and, since $\Gamma(t)_j$ is a closed curve enclosing only the eigenvalue $\lambda(t)_j$, all the elements but those with $\lambda(t)_k = \lambda(t)_j$ of the matrix vanish; for $\lambda(t)_k = \lambda(t)_j$ we have $I(\Gamma(t)_j,\, \lambda(t)_k) = 1$.

Therefore

$$P(t)_j u = \sum_{l=1}^{m_j} u_{jl}E(t)_{jl}, \quad u = \sum_{k\in\mathbb{N}} u_k E(t)_k;$$

where $E(t)_{jl}$, $l = 1,\, \dots,\, m_j$ are the eigenfunctions associated with the eigenvalue $\lambda(t)_j$.

A given coordinate subspace $\mathcal{L}_t \subseteq L^2(M,\, \mu(t))$ may be written as the spanning $\mathcal{L}_t = span\{U(t)\phi(0)_k\}$ where $\phi(0)_k$, $k = 1,\, \dots,\, m$ are eigenfunctions of the Laplacean $\Delta(0)$ on

$L^2(M,\,\mu(0))$, i.e., the transformation function $U(t)$, allows us to identify the coordinate spaces in $L^2(M,\,\mu(t))$ with coordinate spaces in $L^2(M,\,\mu(0))$.

On the other side $\mathcal{L}_t \subseteq L^2(M,\,\mu(t)) = L^2(M,\,\mu(0))$. From now we will see $\mathcal{L}_t$ as a subspace in $L^2(M,\,\mu(0))$ and; $\Delta(t)$ and $D(t)_{f(t)_j}$ as operators in $L^2(M,\,\mu(0))$.

Lemma 8.3.2. *Let $\mathcal{L}_0$ be a finite-dimensional coordinate space on $L^2(M,\,\mu(0))$. Under hypothesis 1; for some finite set $F_{\mathcal{L}_0} \subset]0,\,1]$ and all $t \in [0,\,1] \setminus F_{\mathcal{L}_0}$, we have that, starting with $span\{f(t)_j \mid j = 1,\,\ldots,\,s\}$, after a finite number of iterations of applications of*

$$D(t)_{f(t)_j}(\cdot) = \{\Delta(t)^{-1}(\cdot), f(t)_j\}_t + \{\Delta(t)^{-1}(f(t)_j), (\cdot)\}_t$$

we can obtain a set of functions $\{v_1,\,\ldots,\,v_m\} \subseteq L^2(M,\,\mu(0))$ whose projections $\{\Pi_{\mathcal{L}_t} v_r \mid r = 1,\,\ldots,\,m\}$ onto $\mathcal{L}_t$ span all the space $\mathcal{L}_t$.

Here $f(t)_j := U(t)f(0)_j, \quad j = 1,\,\ldots,\,s$, where $f(0)_j$ are the eigenfunctions in the hypothesis 1.

Proof. Write the m-subspace $\mathcal{L}_t$ as $\mathcal{L}_t = span\{U(t)\phi(0)_k \mid k = 1,\,\ldots,\,m\}$ where $\phi(0)_k$ are eigenfunctions of the Laplacean $\Delta(0)$ on $(M,\,\mu(0))$. The eigenfunctions $\phi(t)_k = U(t)\phi(0)_k$ of the operator $\Delta(t)$ are analytic in t. Then also $\frac{\sqrt{\bar{g}(t)}}{\sqrt{\bar{g}(0)}}\phi(t)_k$ are analytic in t and we have that the "projections"

$$v \mapsto Q(t)_k v := \int_M \frac{\sqrt{\bar{g}(t)}}{\sqrt{\bar{g}(0)}}\phi(t)_k v\sqrt{\bar{g}(0)}\,dx^1 \wedge dx^2 = \int_M \phi(t)_k v\sqrt{\bar{g}(t)}\,dx^1 \wedge dx^2$$

are analytic in t.

Therefore, for analytic w, both the expressions $D(t)_{f(t)_j}w(t)$; $Q(t)_k D(t)_{f(t)_j}w(t)$ and; the sum $\sum_{k=1}^m Q(t)_k D(t)_{f(t)_j}w(t)$, are analytic in t.

By the hypothesis 1, after a finite number $\mathcal{I}$ of iterations of $D(0)_{f(0)_j}$ (starting by applying to the elements of subspace $L_0^{\perp,0} = span\{f(0)_1,\,\ldots,\,f(0)_s\}$) we can obtain a subspace $L_0^{\perp,N}$ containing m functions g_k, $k = 1,\,\ldots,\,m$ close to the eigenfunctions $\phi(0)_k$, $k = 1,\,\ldots,\,m$: $|g(0)_k - \phi(0)_k|_{L^2(M,\,\mu(0))} < \epsilon$. For small enough ϵ,[1] the projections of these functions $g(0)_k$ onto $\mathcal{L}_0$ span all the subspace $\mathcal{L}_0$. In other words

$$0 \neq \det[\Pi_{\mathcal{L}_0} g(0)_k] = \det\left[\sum_{r=1}^m Q(0)_r g(0)_k\right].$$

By $[a(k)]$ we denote the matrix whose columns are the vectors $a(k)$, $k = 1,\,\ldots,\,m$.

The respective functions $g(t)_k$, corresponding to the applications of $D(t)_{f(t)_j}$ (and starting by applying to the space $L_t^{\perp,0} = span\{f(t)_1,\,\ldots,\,f(t)_s\}$) are analytic on t. It follows that also $Q(t)_r g(t)_k$ and the determinant $\det[\sum_{r=1}^m Q(t)_r g(t)_k]$ are analytic.

As a consequence we may conclude that with the exception of a finite number of points $t \in F_{\mathcal{L}_0} \subset]0, 1]$ the determinant $\det[\sum_{r=1}^m Q(t)_r g(t)_k]$ is non-vanishing. $\square$

[1] We take the functions $g(0)_k$ from a space $L_0^{\perp,l}$, for smaller ϵ we (can) choose bigger $l_1 > l$.

8.3.1 Proof of theorem 8.3.1

Lemma 8.3.3. *Let us consider the Navier-Stokes equation in a simply-connected compact manifold M (with C^∞ boundary). Fix a finite set $\{h_1, \ldots, h_d\}$ of vorticities (not necessarily $l^\perp$-saturating) and compute the iteration procedure for the $l^\perp$-saturating set. Fix a finite-dimensional space $\mathcal{F} \subset L^2(M)$. If for some $m \in \mathbb{N}$ we have that the orthogonal projection $\Pi_\mathcal{F} L^{\perp,m}$ of $L^{\perp,m}$ onto $\mathcal{F}$ is onto; then we can observe solid controllability on the component $\left(\nabla^\perp \cdot\right)^{-1}\mathcal{F}$, by means of controlled forcing taking values on $\left(\nabla^\perp \cdot\right)^{-1}\{h_1, \ldots, h_d\}$.*

Proof. Fix a compact set $K \subset \left(\nabla^\perp \cdot\right)^{-1}\mathcal{F}$ and an initial condition $u_0 \in V$. Then consider another compact set $K_1 \subset \left(\nabla^\perp \cdot\right)^{-1}L^{\perp,m}$. From the study we have done in chapter 3 we know that we may find a family of controls in $span\left(\nabla^\perp \cdot\right)^{-1}\{h_1, \ldots, h_d\}$ such that the end-point map (mapping the control into the projection of the final point onto $\left(\nabla^\perp \cdot\right)^{-1}L^{\perp,m}$), together with its small C^0 perturbations cover the set K_1. Moreover the final points u_T are close to points of the form $v_1 + p = (v_1 - P^m v_1) + P^m v_1 + p$ where P^m denotes the orthogonal projection onto $\left(\nabla^\perp \cdot\right)^{-1}L^{\perp,m}$ and; the norm $\|v_1\|$ of v_1 depends only on u_0 and T.

Set the compact K_1 such that $\Pi_\mathcal{F} K_1 \supset \tilde{K} := K + \{x \in \mathcal{F} \mid |x| \leq |v_1|\}$; the family $\Pi_\mathcal{F}(P^m v_1 + p)$ covers $\tilde{K}$ because, the family $P^m v_1 + p$ covers K_1; then the family $\Pi_\mathcal{F} u_T = \Pi_\mathcal{F}(v_1 - P^m v_1) + \Pi_\mathcal{F}(P^m v_1 + p)$ will cover K. We may conclude that the map sending the same controls into the projection of the final point onto $\left(\nabla^\perp \cdot\right)^{-1}\mathcal{F}$, together with its small continuous perturbations, do cover the compact K. □

Proof of theorem 8.3.1. We run over all the finite-dimensional coordinate spaces $\mathcal{L}_0^q$, $q \in \mathbb{N}$ (countable number) of $L^2(M, \mu(0))$. For all metrics $\mu(t)$, with

$$t \in \mathcal{R}_\mu :=]0, 1] \setminus \bigcup_{p\in\mathbb{N}} F_{\mathcal{L}_0^p} = \bigcap_{p\in\mathbb{N}}]0, 1] \setminus F_{\mathcal{L}_0^p}$$

where $F_{\mathcal{L}_0^p} \subset]0,1]$ is the finite set given by lemma 8.3.2; we have that for any $p \in \mathbb{N}$ after a finite number of $\mathcal{I}(p)$ iterations of the operators $D(t)_{f(t)_j}(\cdot) = \{\Delta(t)^{-1}(\cdot), f(t)_j\}_t + \{\Delta(t)^{-1}(f(t)_j), \cdot\}_t$ we obtain a set of functions whose projections onto $\mathcal{L}_t^p$ span the space $\mathcal{L}_t^p$. Here $f(t)_j := U(t)f(0)_j, \quad j = 1, \ldots, s$, where $f(0)_j$ are the eigenfunctions in the hypothesis 1.

The result follows from lemma 8.3.3. □

Remark 8.3.2. *We may also derive theorem 8.3.1 and lemma 8.3.2 from weaker hypothesis: instead of hypothesis 1 is enough to have*

Hypothesis 2. *There is a finite set $h = \{f(0)_1, f(0)_2, \ldots, f(0)_s\}$ of eigenfunctions of the Laplacean $\Delta(0)$ in $L^2(M, \mu(0))$ such that, for each finite-dimensional coordinate space $\mathcal{L}_0 \subset L^2(M, \mu(0))$, we have that after a finite number of iterations of the operation $D(0)_{f(0)_j}(\cdot) = \{\Delta(0)^{-1}(\cdot), f(0)_j\} + \{\Delta(0)^{-1}(f(0)_j), (\cdot)\}$ we obtain a space that projects onto on $\mathcal{L}$.*

Remark 8.3.3. *Theorem 8.3.1 says that we observe solid controllability on observed coordinate space. Anyway we cannot derive approximate controllability from that. To have approximate controllability, (at least using the method of previous works such as [4, 5, 46]) we need the existence of a saturating set.*

8.4 Corollaries

Corollary 8.4.1. *Let M be either the Torus or the Sphere or the Hemisphere and let $\mu(0)$ be the usual metric in M (induced by the Euclidean metric in $\mathbb{R}^3$). For all metrics $\mu(1)$ on M and any homotopy between $\mu(0)$ and $\mu(1)$, there exist a residual set $\mathcal{R}_\mu \subset [0, 1]$ and s eigenfunctions (modes) $f(t)_1, \ldots, f(t)_s$ of the Laplace operator $\Delta(t)$ on $(M, \mu(t))$ such that the Navier-Stokes system on $(M, \mu(t))$ is controllable on observed coordinate space by means of (controlled) forcing applied to the modes $(\nabla^\perp\cdot){-}1\{f(t)_1, \ldots, f(t)_s\}$.*

The number s of eigenmodes may be taken equal to 4 for the Torus, 5 for the Sphere and 3 for the Hemisphere.

Suppose a $l^\perp$-saturating set $\{f(0)_1, \ldots, f(0)_s\}$ does exist for some simply-connected compact flat domain $\Omega \subseteq \mathbb{R}^2$ with analytic boundary. Let M be another simply-connected flat domain. By the Riemann Mapping theorem we may map D onto M, $f : D \to M$, conformally; moreover since the boundaries are analytic we may extend the conformal map analytically to the boundary.

The Euclidean metric on M reads

$$\mu(1) = |f'z|^2 dz d\overline{z} = |Df_x|^2 (dx^1 \otimes dx^1 + dx^2 \otimes dx^2);$$

since $|Df_x|^2$ is strictly positive in $\overline{D}$, we may rewrite $|Df_x|^2$ as $e^{a(x^1, x^2)}$.

Consider the homotopy $\mu(t) = e^{ta}(dx^1 \otimes dx^1 + dx^2 \otimes dx^2)$ between the Euclidean metric $\mu(0)$ in D and, the metric $\mu(1)$ induced in D by the Euclidean one of M. The metric $\mu(t)$ is locally flat because $\Delta(ta) = t\Delta a = 0$ (see [22, section §.2]).

It is possible to prove that the metrics $\mu(t)$ are isometric to metrics induced by Euclidean metrics in some flat domain M_t, so we have the following:

Corollary 8.4.2. *For all analytic simply-connected flat domains M, there are close domains $\tilde{M}$ where we have solid controllability of the Navier-Stokes system on observed coordinate spaces, by means of controlled forcing taking values in the space of vector fields $(\nabla^\perp\cdot)^{-1}\{f(t)_1, \ldots, f(t)_s\}$.*

Now, consider the usual metric on the Hemisphere $\mathbb{S}^2_+$; this metric induces the metric

$$\mu(0) = \frac{4}{(1 + x_1^2 + x_2^2)^2}\,(dx_1 \otimes dx_1 + dx_2 \otimes dx_2)$$

on the unit disk $\mathbb{D}_1$. This metric is obtained by considering the stereographic projection from the south pole of the Riemannian Sphere $\mathbb{S}^2$ (see [22, section §.2]).

Corollary 8.4.3. *For all metrics $\mu(1)$ on the unit disk $\mathbb{D}_1$, and any homotopy $\mu(t)$ between $\mu(0)$ and $\mu(1)$ there exist a residual set $\mathcal{R}_\mu \subset [0, 1]$ and 3 eigenfunctions (modes) $f(t)_1, \ldots, f(t)_s$ of the Laplace operator $\Delta(t)$ on $(M, \mu(t))$ such that the Navier-Stokes system on $(M, \mu(t))$ is controllable on observed coordinate space by means of (controlled) forcing applied to the modes $(\nabla^\perp\cdot){-}1\{f(t)_1, \ldots, f(t)_s\}$.*

The last corollary does not guarantee that we have the controllability result for the Euclidean metric in the unit disk. We can anyway find a homotopy such that the metrics $\mu(t)$ correspond to usual metrics in pieces of spheres in $\mathbb{R}^3$ with radius $\frac{1}{1-t}$, so with constant Gaussian curvature $(1-t)^2$: consider the unit disk $\mathbb{D}_1 \subset \mathbb{R}^2 = \{(x, y, 0) \in \mathbb{R}^3 \mid x^2 + y^2 \leq 1\}$; for each $R \geq 1$ consider a sphere $\mathbb{S}^2(R)$ of radius R and centered at $(0,\, 0,\, 1-R)$; consider the pieces M_R of these spheres, in the half-space $\{(0,\, 0,\, z) \mid z \geq 0\}$, containing the points such that the segment (of the stereographic projection) connecting this point to the south pole $(0,\, 0,\, -2R+1)$ intersects the disk $\mathbb{D}_1$.

Thus we have a family of compact simply-connected manifolds M_R "connecting" the Hemisphere M_1 to the disk $D = \{(x,\, y,\, 1) \in \mathbb{R}^3 \mid x^2 + y^2 \leq 1\}$.

Stereographic projection from the south pole of $\mathbb{S}^2(R)$ maps a disk (laying in the plane $\{(0,\, 0,\, z) \mid z = 1-R\}$) of radius $\frac{R}{2R-1}$ onto the piece M_R; for coordinates $(s_1,\, s_2)$ in that disk the metric (inherited from the Euclidean one in $\mathbb{R}^3$) reads

$$\frac{4R^4}{(R^2+s_1^2+s_2^2)^2}(ds_1 \otimes ds_1 + ds_2 \otimes ds_2).$$

Change the coordinates $(s_1,\, s_2)$ by $(x,\, y) = \frac{2R-1}{R}(s_1,\, s_2)$ in the disk $\mathbb{D}_1$; $(x,\, y)$ are the point in the segment of stereographic projection corresponding to $(s_1,\, s_2)$. The metric in the coordinates $(x,\, y)$ reads

$$\frac{4\left(\frac{R}{2R-1}\right)^2 R^4}{\left(R^2+\left(\frac{R}{2R-1}\right)^2(x^2+y^2)\right)^2}(dx \otimes dx + dy \otimes dy),$$

which may be simplified to

$$\frac{4(2R-1)^2R^2}{(1+4(R-1)R+x^2+y^2)^2}(dx \otimes dx + dy \otimes dy).$$

Making the change of variables $t = 1 - \frac{1}{R}$, for the coefficient $\frac{4(2R-1)^2R^4}{(1+4(R-1)R+x^2+y^2)^2}$, we obtain

$$\frac{4(1+t)^2}{(1+x^2+y^2+t(2+t+(t-2)(x^2+y^2)))^2}$$
$$= \frac{4(1+t)^2}{((1+x^2+y^2)t^2+(2-2x^2-2y^2)t+1+x^2+y^2)^2}.$$

The sum $(1+x^2+y^2)t^2+(2-2x^2-2y^2)t+1+x^2+y^2$, for fixed $(x,\, y)$, attains its minimum at $t = -\frac{1-x^2-y^2}{1+x^2+y^2}$ and the minimum is given by $4 - \frac{4}{1+x^2+y^2}$; as we see this minimum is non-negative and it vanishes only for $(x,\, y) = (0,\, 0)$ which corresponds to $t = -1$.

Therefore for $t \in [0,\, 1]$ the denominator never vanishes and we conclude that the homotopy

$$\mu(t) := \frac{4(1+t)^2}{((1+x^2+y^2)t^2+(2-2x^2-2y^2)t+1+x^2+y^2)^2}(dx \otimes dx + dy \otimes dy)$$

between the metric $\mu(0)$ induced on the disk $\mathbb{D}_1$ by the usual metric of the Hemisphere and, the Euclidean metric on $\mathbb{D}_1$ is analytic on $[0,\, 1]$. Note that, for a fixed pair $(x,\, y)$ the the denominator increases when $t \in [0,\, 1]$ increases. So, the coefficients are bounded below by $\frac{4}{2^2} = 1$ and above by $\frac{16}{(1+x^2+y^2)^2} \geq 4$.

Recall that the metric $\mu(t)$, as constructed, is the metric induced on the disk $\mathbb{D}_1$ by the Euclidean metric on $M_R =: M^t$ which Gaussian curvature is $\frac{1}{R^2} = (1-t)^2$. At $t = 1$ we have the Euclidean metric.

Therefore we have that

Corollary 8.4.4. *There are t close to 1 such that we have solid controllability on observed projection of the Navier-Stokes system on M^t.*

Once more, we can not guarantee solid controllability on observed projection of the equation for the Euclidean metric.

Conclusion and future work

In a few words the method we use reduces controllability on observed component and approximate controllability to the existence of a V-saturating set. It is based on the properties of the bilinear operator and uses techniques from Geometric Control Theory and Lie Algebra Theory.

Associated to a V-saturating set $g = \{g_1, \ldots, g_s\}$ there is an increasing sequence $(G^j)_{j\in\mathbb{N}_0}$ of finite-dimensional subspaces of V such that $G^0 = span\{g\}$ and $\overline{\cup_{i=0}^{+\infty} G^j} = H$.

Given two vector fields u_0, u_1, for big enough N we may drive the equation from u_0 to a neighborhood of u_1 by means of an essentially bounded control taking values in G^N.

Using the continuity of the solution of the equation when the control varies in relaxation metric we may replace the control taking values in G^N by a piecewise constant control. The dynamics of a constant control in G^N may be "imitated" by the dynamics of a fast oscillating essentially bounded control, like $w\sin(wt)a$ taking values in G^{N-1} ($a \in G^{N-1}$); at the final time T we arrive to close end-points (in intermediate instants of time the solutions of the equation may be far from each other). Actually the notion of saturating set was defined to make this imitation possible.

This is not the end of the story, there are some questions to be answered in future works.

As we have seen, under Lions boundary conditions and for controllability on observed coordinate space we need the existence of a "saturating set on coordinate projections" formed by eigenfunctions of the Laplacean, i.e., there is a finite number s of eigenfunctions of the Laplacean such that, for given a N-dimensional coordinate space, after a finite number $\mathcal{I}$ of iterations of the l-saturating procedure we obtain N vectors whose projections onto the given coordinate space, are linearly independent. Clearly a l-saturating set is also saturating on coordinate projections.

We may transfer controllability on observed coordinate space from one simply-connected analytic plane domain Ω to many other plane domains. For that we need Ω to be analytic and to have the existence of a "saturating set on coordinate projections". How to find such an example? A possibility is, perhaps trough the perturbation of the boundary of the rectangle. Consider the Navier-Stokes equation, under Lions boundary conditions, on a rectangle $R :=]0, a[\times]0, b[$ in the plane whose side lengths satisfy the relation $\frac{a^2}{b^2} \in \mathbb{R} \setminus \mathbb{Q}$, i.e., the quotient is an irrational real number. In this case we guarantee, for simplicity, that all the eigenvalues $\lambda = \pi^2\left(\frac{n_1^2}{a^2} + \frac{n_2^2}{b^2}\right)$ in the domain $D(A)$, of the Laplace-de Rham operator $A = \Delta = -\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}$ are simple.

For an analytic domain $\Omega_n \subset R$ close enough to R, order the (repeated) eigenvalues of the Laplacean on Ω and on R, in the space of functions (vorticities) vanishing at the boundary; it is known (see [21] vol. I, section VI.2.1) that the i-th eigenvalue $\lambda_{n,i}$ of Δ on Ω_n converge to the i-th eigenvalue λ_i of Δ relative to the domain R. In particular we may suppose that the first N eigenvalues of the Laplacean on Ω_n are simple.

For the first N eigenfunctions we also know that $|w_{n,i} - w_i|_{C^0(R)}$ is small (for Ω_n close enough to R), where $w_{n,i}$ and w_i are, respectively, the i^{th} the eigenfunction of the Laplacean on Ω_n and R vanishing on the boundary: $\Delta w_{n,i} = \lambda_{n,i} w_{n,i}$ in Ω_n, $w_{n,i} = 0$ on $\partial\Omega_n$; $\Delta w_i = \lambda_i w_i$ in R, $w_i = 0$ on ∂R.

All the eigenfunctions are supposed to be normalized: $|w_{n,i}|_{L^2(R)} = |w_i|_{L^2(R)}$. We identify $w_{n,i}$ with its extension by 0 outside Ω_n.

A precise way to perturb the boundary of the rectangle is using a conformal map from the unit disk $\mathbb{D}_1 \subseteq \mathbb{R}^2$ onto the Rectangle and writing the metric of the Rectangle in the coordinates of the unit disk. Then we perturb this metric in such a way that the perturbed metrics correspond to analytic plane domains close to the Rectangle.

The elliptic integral

$$h(x) = \int_0^x \frac{ds}{\sqrt{(1-s^2)(1-k^2s^2)}},$$

where k is a given real number in $]0,\, 1[$, sends the upper half-plane $\{(x_1,\, x_2) \in \mathbb{R}^2 \mid x_2 \geq 0\} \cup \{\infty\}$ onto the Rectangle $[-a,\, a] \times [0,\, b]$; $a = \int_0^1 \frac{ds}{\sqrt{(1-s^2)(1-k^2s^2)}}$, $b = \int_1^{\frac{1}{k}} \frac{ds}{\sqrt{(1-s^2)(1-k^2s^2)}}$. The map h is analytic in all points $z = x + iy$ in the upper half-plane except in the points ± 1, $\pm\frac{1}{k}$ that correspond to the corners $\pm a$, $\pm a + bi$ of the rectangle; in these four points h is continuous. See for example [7, ch. 8].

The map $g(y) = i\frac{i+y}{i-y}$ maps the unit disk onto the upper half-plane and is analytic in all points of the unit disk. The composition $f = h \circ g$ maps the disk onto the Rectangle and is analytic in all points of the closed unit disk except in the points ± 1, $\frac{\pm 2k + i(1-k^2)}{k^2+1}$, that correspond to the corners $\pm a$, $\pm a + bi$ of the Rectangle.

For the map $f(y)$ we find the expression

$$f(y) = \int_{-i}^{y} \frac{-\frac{2}{(-i+z)^2}dz}{\sqrt{2\frac{(-1+z^2)((-i+z)^2+k^2(i+z)^2)}{(-i+z)^4}}}; \qquad f'(z) = \frac{-\frac{2}{(-i+z)^2}}{\sqrt{2\frac{(-1+z^2)((-i+z)^2+k^2(i+z)^2)}{(-i+z)^4}}}$$

and;

$$|f'(z)|^2 = \frac{2}{|(-1+z^2)((-i+z)^2+k^2(i+z)^2)|}.$$

Now consider the family of mappings $f_t(z) := f(tz)$, with $t \in [\frac{1}{2},\, 1[$ mapping the unit disk onto an analytic compact domain contained in the interior of the Rectangle. These mappings are analytic in all the points in the closure of the unit disk.

For $f_t'(z)$ we find the expression $tf'(tz)$, i.e.,

$$|f_t'(z)|^2 = \frac{2t}{|(-1+t^2z^2)((-i+tz)^2+k^2(i+tz)^2)|};$$

writing $z = x + iy$ we find that

$$|f_t'(z)|^2 = |Df_{t|_{(x,\,y)}}|^2 = \frac{2t}{\sqrt{P(x,\,y,\,k,\,t)}},$$

where $P(x, y, k, t)$ is a polynomial in x, y, k, t. Precisely we have

$$P(x, y, k, t) = \sum_{j=0}^{8} D_j t^j; \qquad D_k = D_k(x, y, k),\ k = 0, \dots, 8;$$

and

$$\begin{aligned}
&D_0 = (1 + k^2)^2; \qquad D_1 = 4(k^4 - 1)y;\\
&D_2 = 8(-2k^2x^2 + (1 + k^4)y^2); \qquad D_3 = -4(k^4 - 1)(x^2 - 3y^2)y;\\
&D_4 = -2x^4(1 - 14k^2 + k^4) - 4x^2y^2(1 + k^2)^2 + 2y^4(7 - 2k^2 + 7k^4);\\
&D_5 = -4(k^4 - 1)(x^2 - 3y^2)(x^2 + y^2)y;\\
&D_6 = 8(x^2 + y^2)^2(-2k^2x^2 + (1 + k^4)y^2);\\
&D_7 = 4(k^4 - 1)(x^2 + y^2)^3y; \qquad D_8 = (1 + k^2)^2(x^2 + y^2)^4.
\end{aligned}$$

The family of mappings f_t, induces the family of metrics

$$\mu(t) = \frac{2t}{\sqrt{P(x, y, k, t)}}(dx \otimes dx + dy \otimes dy)$$

in the unit disk $\mathbb{D}_1$. The corresponding Laplaceans are given by

$$\Delta(t) = \frac{\sqrt{P(x, y, k, t)}}{2t}\Delta(\cdot)$$

where Δ is the Euclidean Laplacean in the disk.

So we have not so bad looking expressions for the Laplacean $\Delta(t)$ and metric $\mu(t)$ that are analytic in $[\frac{1}{2}, 1[$. Moreover

- a finite system of eigenvalues $\lambda(t)_j$ and eigenfunctions $\phi(t)_j$ of $\Delta(t)$ vary analytically in $[1/2, 1[$;
- the same finite system of eigenvalues and eigenfunctions are continuous in 1, because we have a sequence of analytic domains Ω_t converging, as $t \to 1$, in the inclusion sense to R.
- the operation $\{\Delta(t)^{-1}\phi(t)_k, (\cdot)\}_t + \{\Delta(t)^{-1}(\cdot), \phi(t)_k\}_t$ is analytic in $t \in [1/2, 1[$;
- the projections

$$Q_r\left(\{\Delta(t)^{-1}\phi(t)_k, (\cdot)\}_t + \{\Delta(t)^{-1}(\cdot), \phi(t)_k\}_t\right) = \sum_{p=1}^{r} u(t)\phi(t)_{k(p)},$$

  onto a given coordinate space $span\{\phi(t)_{k(p)} \mid p = 1, \dots, r\}$, are also analytic in $[1/2, 1[$.

For the case of the Rectangle we know a saturating set $\{\phi(1)_k \mid, k = 1, \dots, 8\}$ composed by 8 eigenfunctions of the Laplacean $\Delta(1)$. Then we know that for $t = 1$, given a coordinate space $\mathcal{S}(1) = span\{\phi(1)_{s(j)} \mid i = 1, \dots, N\}$, after some iterations of $\{\Delta(1)^{-1}\phi(1)_k, (\cdot)\} + \{\Delta(1)^{-1}(\cdot), \phi(1)_k\}$ (starting by applying to the span of the saturating set), we obtain a set of functions $\{F(1)_1, \ldots, F(1)_N\}$ such that $span\{F(1)_1, \ldots, F(1)_N\}$ projects onto on $\mathcal{S}(1)$; considering the respective iterations for $t \in [1/2, 1[$ (starting by applying to $span\{\phi(t)_k \mid, k = 1, \ldots, 8\}$) we obtain a space $span\{F(t)_1, \ldots, F(t)_N\}$; is it true that this space projects onto the corresponding coordinate space $\mathcal{S}(t)$?

We note that $\lambda(t)_i = (\Delta(t)\phi(t)_i, \phi(t)_i)_{L^2(\mathbb{D}_1, \mu(t))} = (\Delta(1)\phi(t)_i, \phi(t)_i)_{L^2(\mathbb{D}_1, \mu(1))}$, and by the convergence of the eigenvalues $\lambda(t)_i \to \lambda(1)_i$ we derive that

$$|\nabla_1\phi(t)_i|^2_{L^2(\mathbb{D}_1,\mu(1))} = (\Delta(1)\phi(t)_i, \phi(t)_i)_{L^2(\mathbb{D}_1, \mu(1))}$$

converges to

$$|\nabla_1\phi(1)_i|^2_{L^2(\mathbb{D}_1,\mu(1))} = (\Delta(1)\phi(1)_i, \phi(1)_i)_{L^2(\mathbb{D}_1,\mu(1))} = \lambda(1)_i.$$

Therefore we have the convergence of $\phi(t)_i$ to $\phi(1)_i$ in $H^1(\mathbb{D}_1, \mu(1))$. On the other side the operation $\{\Delta(t)^{-1}\phi(t)_k, (\cdot)\}_t + \{\Delta(t)^{-1}(\cdot), \phi(t)_k\}_t$ may be rewritten as

$$D(t)_k(\cdot) = \sqrt{\frac{\bar{g}(1)}{\bar{g}(t)}}\lambda(t)_i^{-1}\left\{\phi(t)_k, (\cdot)\right\}_1 + \sqrt{\frac{\bar{g}(1)}{\bar{g}(t)}}\left\{\Delta(1)^{-1}\left(\sqrt{\frac{\bar{g}(t)}{\bar{g}(1)}} \times (\cdot)\right), \phi(t)_k\right\}_1;$$

since the gradient $\nabla_1\phi(t)_k$ go to $\nabla_1\phi(1)_k$, the idea is to find suitable norms such that at each step we may also take the limits in $\nabla_1\Delta(1)^{-1}\left(\sqrt{\frac{\bar{g}(t)}{\bar{g}(1)}} \times (\cdot)\right)$; $(\cdot)$ and; in the full expression $D(t)_k(\cdot)$. It seems to be not straightforward (at least to the author).

Actually we do not need the convergence of the expressions $D(t)_k(\cdot)$, we simply need to prove that the projections in a specific coordinate space can not be identically zero (for $t \in [1/2, 1[$).

Another question, that it is not so clear, concerns the case of no-slip boundary conditions. In the iterations of the V-saturating procedure we compute some images Bu of elements $u \in V$ by the bilinear operator. What are the conditions $u \in V$ must satisfy in order to still have $Bu \in V$? We recall that the elements of V vanish on the boundary of the domain, then also $\nabla_u^1 u$ vanish on the boundary; when does the projection Bu also vanish on the boundary? The following example shows that the projection of a no-slip vector field, onto the space of divergence free vector fields tangent to the boundary, is not necessarily no-slip:

Example 8.4.1. *Consider the case our domain is the Rectangle $R =]0, \pi[\times]0, \pi[$. Let u be the vector field $u = \begin{pmatrix} -4\sin(2x_1)\cos(4x_2) \\ 2\cos(2x_1)\sin(4x_2) \end{pmatrix} - \begin{pmatrix} -2\sin(4x_1)\cos(2x_2) \\ 4\cos(4x_1)\sin(2x_2) \end{pmatrix}$ and φ the scalar function $\varphi = 2\cos(2x_1)\cos(2x_2) - \frac{1}{2}\cos(4x_1)\cos(4x_2)$, which gradient is given by the expression $\nabla\varphi = \begin{pmatrix} -4\sin(2x_1)\cos(2x_2) + 2\sin(4x_1)\cos(4x_2) \\ -4\cos(2x_1)\sin(2x_2) + 2\cos(4x_1)\sin(4x_2) \end{pmatrix}$.*

The vector field $v = u - \nabla\varphi$ vanishes on the boundary of R, while its projection $P^\nabla v = u$ does not.

More directions may be to adapt our method either to the case of boundary control or to the case of other types of nonlinear equations.

We refer that the method may be applied to the stochastic case, in particular for questions concerning ergodicity of the equation (see [39]) and, density of finite-dimensional projections of distributions (see [2]).

Bibliography

[1] S. Agmon, *Lectures on elliptic boundary value problems*, van Nostrand, 1965. {62}

[2] A. A. Agrachev, S. Kuksin, A. V. Sarychev, and A. Shirikyan, *On finite-dimensional projections of distributions for solutions of randomly forced 2D Navier-Stokes equations*, Ann. Inst. H. Poincaré Probab. Statist. **43** (2007), 399–415. {142}

[3] A. A. Agrachev and Y. L. Sachkov, *Control theory from the geometric viewpoint*, Encyclopaedia of Mathematical Sciences, no. 87, Springer, 2004. {2,81,121}

[4] A. A. Agrachev and A. V. Sarychev, *Navier-Stokes equations: Controllability by means of low modes forcing*, J. math. fluid mech. **7** (2005), 108–152. {2,27,77,85,121,135}

[5] ———, *Controllability of 2D Euler and Navier-Stokes equations by degenerate forcing*, Commun. Math. Phys. **265** (2006), 673–697. {2,3,85,135}

[6] ———, *Solid controllability in fluid dynamics*, Instability in Models Connected with Fluid Flow (C. Bardos and A. Fursikov, eds.), International Mathematical Series, Springer, 2007(to appear), [see arXive:math.OC/0701818v1 — 28 Jan 2007]. {3,77,90,129}

[7] N. I. Akhiezer, *Elements of the theory of elliptic functions*, Translations of Mathematical Monographs, no. 79, AMS, 1990. {140}

[8] V. I. Arnold and B. A. Khesin, *Topological methods in hydrodynamics*, AMS, no. 125, Springer, 1998. {73}

[9] V. I. Arnold, *Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits*, Ann. Inst. Fourier **16** (1966), no. 1, 319–361. {76}

[10] ———, *Lectures on partial differential equations*, Univesitext, Springer, 2004. {91,94}

[11] J.-P. Aubin, *Applied functional analysis*, John Wiley & Sons, 1979. {6}

[12] T. Aubin, *Nonlinear analysis on manifolds. Monge-Ampère equations*, Grundlehren der Mathematischen Wissenschaften, no. 252, Springer, 1982. {70}

[13] A. V. Babin and M. I. Vishik, *Attractors of evolution equations*, Studies in Mathematics and its Applications, no. 25, North-Holland, 1992. {1}

[14] H. Brezis, *Analyse fonctionnelle, théorie et applications*, Masson, 1993. {9,10}

[15] H. Cartan, *Théorie élémentaire des fonctions analytiques d'un ou plusieures variables complexes*, Einsegnement des sciences, Hermann Paris, 1961. {130,133}

[16] ______, *Formes différentielles*, Collection Méthodes, Hermann Paris, 1967. {79,90}

[17] P. Cherrier, *Problèmes de Neumann non linéares sur les variétés Riemanniennes*, J. Funct. Anal. **57** (1984), 154–206. {74}

[18] T. Clopeau, A. Mikelić, and R. Robert, *On the vanishing viscosity limit for the 2D incompressible Navier-Stokes equations with the friction type boundary conditions*, Nonlinearity **11** (1998), 1625–1636. {63}

[19] P. Constantin and C. Foias, *Navier-Stokes equations*, Chicago Lectures in Mathematics, The University of Chicago Press, 1988. {1}

[20] J.-M. Coron and A. V. Fursikov, *Global exact controllability of the Navier-Stokes equations on a manifold without boundary*, Russ. J. Math. Phys. **4** (1996), no. 4, 429–448. {2}

[21] R. Courant and D. Hilbert, *Methods of mathematical physics*, vol. I & II, John Wiley & Sons, 1953. {139}

[22] B. A. Dubrovin, A. T. Fomenko, and S.P. Novikov, *Modern geometry — methods and applications. Part I*, GTM, no. 93, Springer Verlag, 1984. {136}

[23] W. E and J. C. Mattingly, *Ergodicity for the Navier-Stokes equation with degenerate random forcing: Finite dimensional approximation*, Comm. Pure Appl. Math. **54** (2001), 1386–1402. {125}

[24] C. Foias, O. Manley, R. Rosa, and R. Temam, *Navier-Stokes equations and turbulence*, Encyclopedia of Mathematics and its Applications, Cambridge university Press, 2001. {1,61}

[25] I. Fonseca and W. Gangbo, *Degree theory in analysis and applications*, Oxford Lectures Series in Mathematics and its Applications, Oxford University Press, 1995. {48}

[26] A. V. Fursikov and O. Yu. Imanuvilov, *Exact controllability of the Navier-Stokes and Boussinesq equations*, Russian Math. Surveys **54** (1999), no. 3, 565–618. {2}

[27] R. V. Gamkrelidze, *On some extremal problems in the theory of differential equations with applications to the theory of optimal control*, J. SIAM Control **3** (1965), no. 1, 106–128. {35}

[28] ______, *Principles of optimal control theory*, Plenum Press, 1978. {32,35}

[29] P. Grisvard, *Elliptic problems in nonsmooth domains*, Pitman Advanced Publishing Program, 1985. {63}

[30] A. A. Il'in, *The Navier-Stokes and Euler equations on two-dimensional closed manifolds*, Math. USSR-Sb. **69** (1991), no. 2, 559–579. {1,77}

[31] J. Jost, *Riemannian geometry and geometric analysis*, fourth ed., Univesitext, Springer, 2005. {67,69,70,78,79,83}

[32] V. Jurdjević, *Geometric control theory*, Cambridge Studies in Advanced Mathematics, no. 51, Cambridge University Press, 1997. {2,121,123,126}

[33] T. Kato, *Perturbation theory for linear operators*, Classics in Mathematics, Springer, 1995. {131,132}

[34] J. P. Kelliher, *Navier-Stokes equations with Navier boundary conditions for a bounded domain in the plane*, SIAM J. Math. Anal. **38** (2006), no. 1, 210–232. {63}

[35] J.-L. Lions and E. Magenes, *Non-homogeneous boundary value problems and applications*, vol. I, Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen, no. 181, Springer-Verlag, 1972. {10,62}

[36] J.-L. Lions and E. Zuazua, *Contrôllabilité exacte des approximations de Galerkin des équations de Navier-Stokes*, C. R. Acad. Sci. Paris **324** (1997), 1015–1021. {2}

[37] ———, *Exact boundary controllability of Galerkin approximations of Navier-Stokes equations*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **26** (1998), 605–621. {2}

[38] C. Marchioro and M. Pulvirenti, *Mathematical theory of incompressible nonviscous fluids*, Applied Mathematical Sciences, no. 96, Springer-Verlag, 1994. {65}

[39] J. C. Mattingly and É. Pardoux, *Malliavin calculus for the stochastic 2D Navier-Stokes equation*, Comm. Pure Appl. Math. **59** (2006), no. 12, 1742–1790. {142}

[40] J. Nečas, *Les méthodes directes en théorie des équations elliptiques*, Masson & Cie Éditeurs, 1967. {54}

[41] J. Peetre, *Espaces d'interpolation et théorème de Soboleff*, Ann. Inst. Fourier (1978), 279–317. {62}

[42] V. Priebe, *Solvability of the Navier-Stokes equations on manifolds with boundary*, Manuscripta Math. **83** (1994), 145–159. {1,70}

[43] M. Renardy and R. C. Rogers, *An introduction to partial differential equations*, Texts in Applied Mathematics, no. 13, Springer-Verlag, 1993. {9}

[44] S. S. Rodrigues, *Controllability issues for the Navier-Stokes equation on a Rectangle*, Proceedings 44th IEEE CDC-ECC'05(CD-ROM), December 2005, pp. 2083–2085. {3}

[45] ———, *Navier-Stokes equation on a plane bounded domain: Continuity properties for controllability*, Taming Heterogeneity and Complexity of Embedded Control (F. Lamnabhi-Lagarrigue, S. Laghrouche, A. Loria, and E. Panteley, eds.), ISTE Ltd, 2006, Proc. CTS-HYCON Workshop on Nonlinear and Hybrid Control, ch. 33, pp. 585–615. {3,18}

[46] ———, *Navier-Stokes equation on the Rectangle: Controllability by means of low modes forcing*, J. Dyn. Control Syst. **12** (2006), no. 4, 517–562. {3,27,96,135}

[47] ———, *Controllability of nonlinear pde's on compact Riemannian manifolds*, Proceedings WMCTF'07(CD-ROM), April 2007, pp. 462–493. {3}

[48] ———, *Navier-Stokes equation on the Rectangle*, Preprint SISSA 23/2005/M, Apr 2005. {3,18,20,36,37,48,96}

[49] ———, *2D Navier-Stokes equation: Continuity properties for controllability*, Preprint SISSA 30/2006/M, Jun 2006. {3}

[50] S. Rosenberg, *The Laplacian on a Riemannian manifold*, Student texts, no. 31, London Mathematical Society, 1997. {68,70}

[51] A. Shirikyan, *Approximate controllability of three-dimensional Navier-Stokes equations*, Comm. Math. Phys. **266** (2006), 123–151. {2,3}

[52] ———, *Exact controllability in projections for three-dimensional Navier-Stokes equations*, Ann. I. H. Poincaré – AN (to appear). {2,3}

[53] R. S. Strichartz, *A guide to distribution theory and Fourier transformations*, World Scientific, 2003. {14}

[54] R. Temam, *Navier-Stokes equations and nonlinear functional analysis*, second ed., CBMS-NSF Regional Conference Series in Applied Mathematics, Society for Industrial and Applied Mathematics, 1995. {8,62}

[55] ———, *Infinite-dimensional dynamical systems in mechanics and physics*, second ed., Applied Mathematical Sciences, no. 68, Springer, 1997. {8}

[56] ———, *Navier-Stokes equations: Theory and numerical analysis*, AMS Chelsea Publishing, 2001. {1,9,10,11,17,53,54,56,58,61,62}

[57] A. Trautman, *Differential geometry for physicists* (stony brook lectures), Monographs and textbooks in physical science, Bibliopolis, 1984. {78,79}

Index